The University of Adelaide
School of Economics

Research Paper No. 2010-11
May 2010

Closed-Form Likelihood Expansions for Multivariate Time-Inhomogeneous Diffusions

Seungmoon Choi

Closed-Form Likelihood Expansions for Multivariate Time-Inhomogeneous Diffusions

Seungmoon Choi*

School of Economics, University of Adelaide

January, 2010

Abstract

The aim of this paper is to find approximate log-transition density functions for multivariate time-inhomogeneous diffusions in closed form. There are many empirical evidences that the underlying data generating processes for many economic variables might change over time. One possible way to explain the time-dependent behavior of state variables is to model the drift or volatility terms as functions of time t as well as state variables. Closed-form likelihood expansions for multivariate time-homogeneous diffusions have been obtained by Aït-Sahalia (2008). This research is built on his work and extends his results to time-inhomogeneous cases. Simulation study reveals that our method yields a very accurate approximate likelihood function that can be a good candidate when the true likelihood function is unavailable.

KEY WORDS: Likelihood function; Multivariate time-inhomogeneous diffusion; Reducible diffusions, Irreducible diffusions

1 Introduction

The history of multivariate diffusion models is long (Brennan and Schwartz (1979), Langetieg (1980) and Stambaugh (1988)), whereas univariate diffusion model has been dominantly used in the literature since Merton's seminal work in the 1970s. Stochastic volatility model (Heston (1993), Andersen and Lund (1997) and Gallant and Tauchen (1998)), multifactor affine term-structure model (Duffie and Kan (1996) and Dai and Singleton (2000)), and quadratic term-structure model (Ahn, Dittmar, and Gallant (2002)) are more recent examples.

*I am very grateful to Yacine Aït-Sahalia, Jiti Gao, Chirok Han, Bruce Hansen, Vance Martin, Joon Y. Park, Peter Phillips, Yoon-Jae Whang, Jun Yu and the seminar participants at the NZESG meeting in Auckland, Yonsei University, Korea University, Seoul National University, University of Melbourne, and Queensland University of Technology for helpful suggestions and comments. Contact: Seungmoon Choi, School of Economics, University of Adelaide, North Terrace, SA 5005, Australia, Email: seungmoon.choi@adelaide.edu.au

Continuous-time diffusion models written as m-dimensional stochastic differential equation (SDE) form of (1) have been widely used in many different economic areas from finance to game theory. These fields include optimal control theory (Dixit and Pindyck (1994)), contingent claim pricing and portfolio choice (Duffie (2001)), dynamics of exchange rate (Froot and Obstfeld (1991)), contract theory (Holmstrom and Milgrom (1987)) and game theory (Bolton and Harris (1999)). In these literature, researchers model many economic variables such as interest rates, asset prices, exchange rates, electricity prices, profits, revenue, costs, payoffs and so forth using diffusion processes.

$$dX_t = \mu\left(t, X_t\right) dt + \sigma\left(t, X_t\right) dW_t \tag{1}$$

Equation (1) is a time-inhomogeneous diffusion model since the drift function, μ, or the volatility function, σ, depend on not only state variable vector X_t but also time t. If they are functions of just state variables X_t it is called a time-homogeneous diffusion process.

Naming some evidences of time dependence of economic variables, the volatilities of interest rates at all maturities are known to increase around releases of key macroeconomic announcements and on Federal Open Market Committee (FOMC) meeting days (Bartolini, Bertola, and Prati (2002) and Piazzesi (2005)). The former events also cause increased volatility of exchange rates (Andersen, Bollerslev, Diebold, and Vega (2003)). The return volatilities of stock price, exchange rate and US Treasury bond futures contract, and the electricity price show intraday periodicity (Lockwood and Linn (1990), Bollerslev, Cai, and Song (2000), Andersen and Bollerslev (1997), and Misiorek, Trueck, and Weron (2006)). For the lower frequency, calendar effects of these variables are well documented in the literature (Hansen and Lunde (2005), Jordan and Jordan (1991), Andersen and Bollerslev (1998), and J. and Schwartz (2002)). Phillips (2001) and Franses (1996) discuss importance of direct modeling of trends and seasonalities, respectively. Model (1) has a potential to explain these time-dependent properties of the underlying processes.

Ho and Lee (1986), Hull and White (1990, 1994), and Black and Karasinski (1991) adopt time-inhomogeneous diffusion model for the short-term interest rate to match the current yield curve and volatility structure when pricing bond and bond options. Time-inhomogeneous affine processes have been studied by Filipović (2005). Stock return volatility is dependent on both stock price and time in Rubinstein (1994) and Derman and Kani (1994).

The problem with regard to estimation of diffusion models is that the observed data are discrete while the model specification is in continuous-time. Ignoring the difference is known to result in inconsistent estimators (Merton (1980), Lo (1988) and Melino (1994)). In recent years, a lot of econometric methods have been proposed to estimate continuous-time diffusion models without requiring a continuous record of observations, for example, maximum likelihood estimation (MLE) (Pearson and Sun (1994), Chen and Scott (1993), Pedersen (1995), Santa-Clara (1995) and Durham and Gallant (2002)), simulation method (Gouriéroux, Monfort, and Renault (1993), Duffie and Singleton (1993)), generalized method of moment (GMM) (Hansen and Scheinkman (1995), Duffie and Glynn (2004), Bibby and Sørenson (1995) and Kessler and Sørenson (1999)), efficient method of moment (EMM) (Gallant and Tauchen (1996,1998)), nonparametric method (Aït-Sahalia (1996a, 1996b), Stanton (1997) and Arapis and Gao (2006)) and Bayesian method (Eraker (2001), Elerian, Chib, and Shephard (2001) and Jones (2003)).

MLE method has not been used much because the closed-form transition density function is unknown for most of diffusion processes. Therefore, people have numerically solved Kolmogorov partial differential equation which the transition density function should satisfy or simulated sample paths very finely along which the process is sampled in order to approximate the transition density. Neither of these methods produces a closed-form density function. Aït-Sahalia (2002) has established a way to get an approximate transition density function explicitly using Hermite series expansion for univariate diffusion models. His work is extended to univariate jump diffusions by Schaumburg (2001), to univariate time-inhomogeneous diffusion models by Egorov, Li, and Xu (2003), and Choi (2009) applies his method to regime-switching univariate diffusion models for the short-term interest rate. Closed-form log-likelihood expansions for multivariate diffusions also have been obtained by Aït-Sahalia (2008), which is generalized to multivariate jump diffusions by Yu (2007) and Purzitsky (2003). This article is built on Aït-Sahalia (2008) to extend his results for multivariate time-homogeneous models to time-inhomogeneous cases.

The purpose of this paper is to approximate true but in most of cases unknown transition density function of multivariate time-inhomogeneous diffusion processes in an explicit form. We need mild technical assumptions such as smoothness and linear growth conditions for the drift and volatility functions, for the existence and uniqueness of the solution to SDE (2) and to get a closed-form expansion of the likelihood.

A diffusion model is called reducible if there exists a transformation which can transform the original diffusion process into a unit diffusion whose volatility is an identity matrix. Otherwise it is said to be irreducible. If diffusions are reducible, two reducible methods (Hermite-expansion and Kolmogorov-equation methods) can be used to obtain the approximate log-transition density function. This paper provides an alternative form of the approximate transition density function to Egorov, Li, and Xu (2003) by applying the Kolmogorov-equation method.

As explained in Section 3.1, if we cannot find an antiderivative for the transformation of a reducible diffusion, Kolmogorov-equation method cannot be used whether it is time-homogeneous or time-inhomogeneous. Even so, Hermite-expansion method is still available for time-homogeneous reducible diffusions. However, when there is no explicit function for the derivatives of the transformation with respect to time variable either, none of the reducible methods can be used for time-inhomogeneous reducible diffusion models. Fortunately, the irreducible method introduced in Section 3.2.2 can resolve this problem.

When a multivariate time-homogeneous diffusion is irreducible we can no longer apply any of the reducible methods. The key idea of finding approximate log-transition density of irreducible diffusions is to postulate the form of the log-likelihood expansion as the one attained from the reducible case and use the Kolmogorov equations to get the partial differential equations (PDEs) of each coefficient. Unlike the reducible case, generally these PDEs cannot be solved analytically. Aït-Sahalia (2008) Taylor expands each coefficient of the postulated log-transition function and matches the same order terms in each PDE to get the coefficients of the Taylor-expansions. This must be done successively from low to high order of the Taylor-expansion and from low to high order of the coefficients of the log-transition density because high order terms depend on low order terms. In contrast to the time-homogeneous case, infinitely many terms of the Taylor-expansions of the first two coefficients of the log-likelihood expansion for time-inhomogeneous diffusions cannot be determined from the PDEs. Therefore the recursive way of getting the coefficients breaks down. However, we can show that the approximate log-likelihood function itself does not depend on those indeterminate terms because they are cancelled out in the log-likelihood function although most of the coefficients of Taylor-expansions suffer from indeterminacy problems. In consequent, we can set the indeterminate terms to zero and use the differential equations of the coefficients to establish an approximate log-transition density function of time-inhomogeneous diffusion processes.

Next section introduces our time-inhomogeneous diffusion models and states required assumptions in more detail. Section 3 describes main results of how to find an approximate transition density function for both reducible and irreducible cases. Convergence of the log-likelihood function is discussed in Section 4. Also we conduct Monte Carlo simulation study to see the effectiveness of our new method in Section 5. Then conclusion follows.

2 Model and Assumptions

Consider an m-dimensional multivariate time-inhomogeneous diffusion process,

$$dX_t = \mu\left(t, X_t; \theta\right) dt + \sigma\left(t, X_t; \theta\right) dW_t. \tag{2}$$

X_t is an $m \times 1$ vector of state variables in $S_X \subset R^m$. $\mu\left(t, X_t; \theta\right)$ is an $m \times 1$ vector of drift functions, which are also known as infinitesimal mean or expected infinitesimal displacement, and $\sigma\left(t, X_t; \theta\right)$ is an $m \times m$ volatility (or dispersion) matrix. In our model, $\mu\left(t, X_t; \theta\right)$ or $\sigma\left(t, X_t; \theta\right)$ depend not only the state variables X_t but also time variable t and they are known up to a parameter vector $\theta \in \Theta$, which is a compact subset of R^p. For notational simplicity, we will omit the parameter vector θ until Section 5 where we discuss maximum likelihood estimator of θ. Without loss of generality, W_t is an $m \times 1$ vector of independent Brownian motions. Any correlation structures between the state variables can be modelled by using off-diagonal terms in the dispersion matrix, which needs not be symmetric. Let $S_X \subset R^m$ be the domain of the diffusion X_t. For simplicity, we assume the following for S_X.

Assumption 1. *S_X is a product of m intervals with lower and upper limits, $\underline{x}_i$ and $\bar{x}_i$, where possibly $\underline{x}_i = -\infty$ and/or $\bar{x}_i = \infty$, in which case the intervals are open at infinite limits.*

Characterization of a diffusion depends on the following infinitesimal variance-covariance matrix, $\upsilon(t,x)$ rather than the dispersion matrix, $\sigma(t,x)$.

$$\upsilon(t,x) \equiv \sigma(t,x)\,\sigma^T(t,x), \tag{3}$$

where $\sigma^T(t,x)$ is transposition of $\sigma(t,x)$.

Assumption 2. *The diffusion matrix $\upsilon(t,x)$ is positive definite for all (t,x) in the interior of $[0,\infty)\times S_X$.*

Even though there may be a continuum of choices for $\sigma(t,x)$ satisfying (3), the transition probability of the process is the same for each one of these $\sigma(t,x)$ (see Remark 5.17 and Section 5.3 in Stroock and Varadhan (1979)). If $\mu(t,x) = \mu'(t,x)$ and $\upsilon(t,x) = \upsilon'(t,x)$ for two processes X_t and X_t', two have the same joint distribution function given that X_0 and X_0' have the identical distribution (Friedman (1975) Vol 1. p151).

There are two different notions of solutions to the SDE (2), strong and weak solutions. Loosely speaking, for a strong solution, on a given probability space only X_t is constructed with respect to a filtration generated by a given Wiener process $\{W_t\}$ and a given initial random variable X_0. In the case of a weak solution, however, not only X_t but also the filtration, the driving Brownian motion and the probability space should be built as a part of the solution. The difference between strong solution and weak solution is very akin to that between a random variable and its distribution (See Zvonkin and Krylov (1981) for more on strong and weak solutions). Hence, it is sufficient to have a weak solution of a SDE (2), when the problem involves only the measure on the space of trajectories such as the existence of a transition density for a process, the determination of various probabilities and mathematical expectations and so forth. On the other hand, for the situations where a particular family of trajectories should be considered, for instance, stochastic optimal controlling problem, we must consider a strong solution. Therefore, the weak solution concept is sufficient in this paper because we focus on getting an approximate transition density function of a diffusion model. Moreover, it is natural to consider a weak solution from the viewpoint of modelling a stochastic process since a weak solution does not required to specify the Wiener process in advance. A strong solution is a weak solution, but the converse is not true in general. So, let us first discuss assumptions for the uniqueness and existence of a strong solution to the SDE (2).

Assumption 3. *For $i,j = 1,2,\cdots,m$, $\mu_i(t,x)$ and $\sigma_{ij}(t,x)$ are infinitely differentiable with respect to $x \in S_X$ and $t \in [0,\infty)$.*

This infinite differentiability of drift and dispersion coefficients is necessary for computing the approximate expansions of the transition density. As a by-product of Assumption 3, the uniqueness of a strong solution to the SDE (2) is also guaranteed. In fact, we only need local Lipschitz continuity of $\mu(t,x)$ and $\sigma(t,x)$ in x; i.e., for every $C > 0$ there exists a constant $K_C > 0$ such that for every $t \in [0,\infty)$, $\|x\| \le C$ and $\|x'\| \le C$:

$$\|\mu(t,x) - \mu(t,x')\| + \|\sigma(t,x) - \sigma(x',t)\| \le K_C\,\|x - x'\|$$

in order to ensure the uniqueness of a strong solution to (2) if it exists (see e.g. Theorem 5.2.5 in Karatzas and Shreve (1998)). The notation $\|\cdot\|$ stands for the usual Euclidean norm. The local Lipschitz continuity can be obtained from once differentiability of the drift and volatility functions in Assumption 3 by applying the mean value theorem of calculus to $\mu_i(t,x)$ and $\sigma_{ij}(t,x)$.

In addition to the local Lipschitz condition, we need further restrictions on the coefficients, $\mu(t,x)$ and $\sigma(t,x)$ for the existence of a non-exploding strong solution to the SDE (2).

Assumption 4. *There is a constant $K>0$ such that for all $(t,x)\in[0,\infty)\times S_X$*

$$\|\mu(t,x)\|^2 \le K\left(1+\|x\|^2\right) \text{ and } \|\sigma(t,x)\|^2 \le K\left(1+\|x\|^2\right) \tag{4}$$

Derivatives of $\mu_i(t,x)$'s and $\sigma_{ij}(t,x)$'s exhibit at most polynomial growth.

It is the behavior of the process near the infinity that is important for the existence of a non-exploding solution to (2). Assumption 4 ensures the existence of a non-exploding solution by keeping the rate of growth of both $\mu(t,x)$ and $\sigma(t,x)$ at most linear in x as $\|x\|$ goes to infinity. In other words, the coefficients must not grow faster than linearly in magnitude for large $\|x\|$, for the process not to explode in finite time. Without the condition (4), we can show the existence and uniqueness of a strong solution to (2). Then the process (2) can reach infinity in finite time with positive probability. The polynomial growth assumption for the derivatives of the drift and dispersion functions simplifies things in view of the fact that the tails of the transition density function decrease exponentially. Assumption 4 is violated for some models, for instance $\mu(t,x)=x-x^3$. We can replace the assumption for $\mu(t,x)$ in Assumption 4 with a weaker condition

$$\sum_{i=1}^{m} x_i\mu_i(t,x) \le K\left(1+\|x\|^2\right)$$

and the same restriction for $\sigma(t,x)$ to cover such a case. This is weaker than (4) because, if $\|\mu(t,x)\|^2 \le K\left(1+\|x\|^2\right)$ then $\sum_{i=1}^{m} x_i\mu_i(t,x) \le \frac{1}{2}\left(\|x\|^2+\|\mu(t,x)\|^2\right) \le \frac{K+1}{2}\left(1+\|x\|^2\right)$. To my best knowledge, little is known about the existence and uniqueness of strong solutions to multivariate SDEs except the cases covered by Assumption 3 and 4.

Turning to a weak solution, Stroock and Varadhan (1969) show that the existence and uniqueness of a weak solution can be verified when $\mu(t,x)$ is bounded, $\sigma(t,x)$ is bounded and continuous, and $\upsilon(t,x)$ is positive definite for all $(t,x)\in[0,\infty)\times S_X$. Boundedness of the coefficients is so restrictive that many of the models in practice do not satisfy it. Skorohod (1965 p 59) proves that there exists a non-exploding weak solution if $\mu(t,x)$ and $\sigma(t,x)$ are continuous with respect to $(t,x)\in[0,\infty)\times S_X$ and satisfy the linear growth condition (4). Therefore, if we can show the uniqueness of a weak solution to (2), we can prove the existence and uniqueness of a weak solution.

There is little theory on the uniqueness of a weak solution (that is, uniqueness in the sense of the probability law or distributional uniqueness) to SDE with unbounded and continuous coefficients. However, the uniqueness of a weak solution can be obtained by proving the third concept of uniqueness, pathwise uniqueness, which is a generalization of the strong solution uniqueness. See Watanabe and Yamada (1971a, b) for more details about pathwise uniqueness. In fact, the pathwise uniqueness implies the uniqueness for both strong and weak solutions. Furthermore, if there exists a weak solution and pathwise uniqueness holds then the existence and uniqueness for both strong and weak solutions are assured (Theorem 10.13 Chung and Williams (1990)). The pathwise uniqueness holds under Assumptions 3 and 4. Hence, any diffusion process satisfying those assumptions has a unique weak solution as well as a unique strong solution.

The Lipschitz condition plays a key role in the proof of the uniqueness of the solution. But, some multivariate diffusion models, for example the extended multivariate Cox-Ingersoll-Ross model with $\sigma(t,x) = diag\left(\sigma_{0i}\sqrt{x_i}e^{\sigma_{1i}t}\right)_{i=1,2,\cdots,m}$, does not meet the Lipschitz condition, not to mention Assumption 3. Here $diag\left(a_i\right)_{i=1,2,\cdots,m}$ is the $m \times m$ diagonal matrix whose diagonal elements are $a_i,\ i = 1, 2, \cdots, m$. Note that x^α is not differentiable at $x = 0$ and does not satisfy a local Lipschitz condition in a neighborhood of $x = 0$ for $0 < \alpha < 1$. Assumption 3 can be relaxed to insure pathwise uniqueness (Watanabe and Yamada (1971a)). So, there is a unique weak and strong solution to the extended multivariate CIR model. Since their work, Sonoc (1998) and Swart (2001) have refined the pathwise uniqueness conditions.

3 Closed-Form Log-Transition Density Expansions

3.1 Univariate Time-Inhomogeneous Diffusions

Let's first review the results for univariate time-inhomogeneous diffusions by Egorov, Li, and Xu (2003) and discuss an alternative approach to find approximate transition density functions and a problem in these two approaches. We need to transform the original diffusion process X_t twice in order to make the transition density of the transformed diffusion process close to the Normal. The transition density function of X_t can be recovered using Jacobian.

First, transform univariate diffusion process, (2) to unit diffusion process, (6) by the following Lamperti transformation from X to Y,

$$y = \gamma(t,x) \equiv \int^x \frac{1}{\sigma(t,\omega)} d\omega. \tag{5}$$

Using Ito's formula,

$$dY_t = \mu_Y(t, Y_t)\, dt + dW_t, \tag{6}$$

and $\mu_Y(t,y) = \frac{\mu\left[\gamma^{inv}(t,y),t\right]}{\sigma\left[\gamma^{inv}(t,y),t\right]} - \frac{1}{2}\frac{\partial\sigma\left[\gamma^{inv}(t,y),t\right]}{\partial x} + \frac{\partial\gamma\left[\gamma^{inv}(t,y),t\right]}{\partial t}$, where $\gamma^{inv}(t,y)$ denotes the inverse of a transformation γ from $x \in R^m$ to $y \in R^m$, $y = \gamma(t,x)$ when it exists. The first transformation makes the tail behavior of p_Y close to Gaussian by making the dispersion term equal to 1. However, p_Y gets concentrated and peaked around the conditional value y_0 when the time difference between two consecutive observations,

$\Delta = t - t_0$, becomes small. Hence, we need the second transformation from Y to Z

$$z \equiv \frac{y - y_0}{\sqrt{\Delta}}.$$

This transformation makes the transition density of Z_t, p_Z close to the Normal. As a consequence, Hermite-expansion of p_Z around standard normal density function, $\phi(z) = \frac{1}{\sqrt{2\pi}} e^{-z^2/2}$,

$$p_Z^{(J)}(t, z|t_0, y_0) \equiv \phi(z) \sum_{j=0}^{J} \eta_Z^{(j)}(t, t_0, y_0) H_j(z) \tag{7}$$

can produce an accurate approximate density with relatively a few coefficients. Hermite polynomials $H_j(z)$'s are defined by $H_j(z) \equiv \phi(z)^{-1} \frac{d^j}{dz^j}\phi(z)$, for example $H_0(z) = 1$, $H_1(z) = -z$, $H_2(z) = z^2 - 1$, $H_3(z) = -z^3 + 3z$, $H_4(z) = z^4 - 6z^2 + 3$, $H_5(z) = -z^5 + 10z^3 - 15z$ and $H_6(z) = z^6 - 15z^4 + 45z^2 - 15$ and these are orthogonal base functions since

$$\int_{-\infty}^{\infty} H_i(z) H_j(z) \phi(z)\, dz = \begin{cases} i! & \text{if } i = j \\ 0 & \text{otherwise} \end{cases}.$$

Using the orthogonality of Hermite polynomials we can obtain the coefficients $\eta^{(j)}(\Delta, y_0, s)$ of the Hermite expansion (7) such that

$$\begin{aligned} \eta_Z^{(j)}(t, t_0, y_0) &= (1/j!) \int_{-\infty}^{\infty} H_j(z)\, p_Z(t, z|t_0, y_0)\, dz \\ &= (1/j!) \int_{-\infty}^{\infty} H_j(z) \sqrt{\Delta} p_Y\left(t, \sqrt{\Delta}z + y_0|t_0, y_0\right) dz \\ &= (1/j!) \int_{-\infty}^{\infty} H_j\left(\frac{y - y_0}{\sqrt{\Delta}}\right) p_Y(t, y|t_0, y_0,)\, dy \\ &= (1/j!)\, E\left[H_j\left(\frac{Y_t - y_0}{\sqrt{\Delta}}\right) \middle| Y_{t_0} = y_0 \right]. \end{aligned}$$

The conditional expectation can be approximated up to any order by Taylor expansion using infinitesimal operator,[1] A_Y defined by

$$A_Y \circ f(t, y, t_0, y_0) = \frac{\partial f(t, y, t_0, y_0)}{\partial t} + \mu_Y(t, y) \frac{\partial f(t, y, t_0, y_0)}{\partial y} + \frac{1}{2} \frac{\partial^2 f(t, y, t_0, y_0)}{\partial y^2}.$$

Therefore, for any infinitely differentiable function f

$$E\left[f(t, Y_t, t_0, Y_{t_0}) \middle| Y_{t_0} = y_0 \right] = \sum_{i=0}^{K} \frac{\Delta^i}{i!} A_Y^i \circ f(t, y, t_0, y_0)\Big|_{y=y_0, t=t_0} + O\left(\Delta^{K+1}\right)$$

[1] In general, if univariate diffusion process X_t follows

$$X_t = \mu(X_t, t; \theta)\, dt + \sigma(X_t, t; \theta)\, dW_t$$

then

$$\begin{aligned} \lim_{\Delta \to 0} \frac{U_\Delta g(X_t, t) - U_0 g(X_t, t)}{\Delta} &= \lim_{\Delta \to 0} \frac{E[g(X_t, t)\,|X_s = x_0] - g(x_0, s)}{\Delta} = A_X \circ g(x, t) \\ &= \frac{\partial g(x, t)}{\partial t} + \mu(x, t; \theta) \frac{\partial g(x, t)}{\partial x} + \frac{1}{2}\sigma^2(x, t; \theta) \frac{\partial^2 g(x, t)}{\partial x^2}\Big|_{x=x_0, t=s} \end{aligned}$$

In this way, Egorov, Li, and Xu (2003) approximate p_Y as

$$p_Y^{(J,K)}(t,y|t_0,y_0) = \Delta^{-\frac{1}{2}}\phi\left(\frac{y-y_0}{\sqrt{\Delta}}\right)\left\{\sum_{j=0}^{J}\frac{1}{j!}\left[\sum_{i=0}^{K}\frac{\Delta^i}{i!}A_Y^i\circ H_j\left(\frac{y-y_0}{\sqrt{\Delta}}\right)\Big|_{y=y_0}\right]H_j\left(\frac{y-y_0}{\sqrt{\Delta}}\right)\right\} \tag{8}$$

That is to say, we have to calculate $\eta_Z^{(j)}(t,,y_0t_0)$'s up to K-th order and truncate (7) at J in order to find the approximation, $p_Y^{(J,K)}$. We will call this Hermite-expansion method.

There is an alternative way to obtain the approximate expansion to p_Y. After letting $J\to\infty$ in $p_Y^{(J,K)}$ if we rearrange all terms in (8) according to powers of Δ, the resulting expansion up to K-th order is given by (9) and its coefficients can be calculated recursively by (10).

$$p_Y^{(K)}(t,y|t_0,y_0) = \Delta^{-\frac{1}{2}}\phi\left(\frac{y-y_0}{\sqrt{\Delta}}\right)\exp\left(\int_{y_0}^{y}\mu_Y(t,\omega)\,d\omega\right)\sum_{k=0}^{K}c_Y^{(k)}(t,y|t_0,y_0)\frac{\Delta^k}{k!}, \tag{9}$$

where $c_Y^{(0)}(t,y|t_0,y_0)=1$ and for all $k\geq 1$

$$\begin{aligned} c_Y^{(k)}(t,y|t_0,y_0) \quad &= \quad k\int_0^1 u^{k-1}\Big\{\lambda_Y(t,y_0+u(y-y_0),y_0)\,c_Y^{(k-1)}(t,y_0+u(y-y_0)|t_0,y_0) \\ &+\frac{1}{2}\frac{\partial^2 c_Y^{(k-1)}(t,\omega|t_0,y_0)}{\partial\omega^2}\Big|_{\omega=y_0+u(y-y_0)} - \frac{\partial c_Y^{(k-1)}(t,y_0+u(y-y_0)|t_0,y_0)}{\partial t}\Big\}du, \end{aligned} \tag{10}$$

where

$$\lambda_Y(t,y,y_0) = -\frac{1}{2}\left[\mu_Y(t,y)^2+\frac{\partial\mu_Y(t,y)}{\partial y}\right]-\int_{y_0}^{y}\frac{\partial\mu_Y(t,\omega)}{\partial t}d\omega.$$

For the case of univariate time-homogeneous model, Aït-Sahalia (1999) verifies that we can get very good approximate expansion with $K=1$ or $K=2$ at most.

The closed-form expansion of the logarithm of the transition density function can be found by Taylor-expanding $\ln\left(\sum_{k=0}^{K}c_Y^{(k)}(t,y|t_0,y_0)\frac{\Delta^k}{k!}\right)$ obtained from $\ln\left[p_Y^{(K)}(t,y|t_0,y_0)\right]$ in Δ rather than just taking logarithm of (9). The K-th order approximate log-likelihood function of (6) is given by

$$l_Y^{(K)}(t,y\mid t_0,y_0) = -\frac{1}{2}\ln(2\pi\Delta)+\frac{C_Y^{(-1)}(t,y\mid t_0,y_0)}{\Delta}+\sum_{k=0}^{K}C_Y^{(k)}(t,y\mid t_0,y_0)\frac{\Delta^k}{k!} \tag{11}$$

and the coefficients are

$$C_Y^{(-1)}(t,y\mid t_0,y_0) = -(y-y_0)^2/2$$

$$C_Y^{(0)}(t,y\mid t_0,y_0) = (y-y_0)\int_0^1\mu_Y(t,y_0+u(y-y_0))\,du.$$

Given $C_Y^{(-1)},C_Y^{(0)},\cdots,C_Y^{(k-1)}$, $C_Y^{(k)}$, $k\geq 1$, can be calculated recursively by

$$C_Y^{(k)}(t,y\mid t_0,y_0) = k\int_0^1 G_Y^{(k)}(t,y_0+u(y-y_0)\mid t_0,y_0)\,u^{k-1}du$$

where

$$G_Y^{(1)}(t,y\mid t_0,y_0) = \lambda_Y(t,y,y_0)$$

and for $k \geq 2$

$$G_Y^{(k)}(t, y \mid t_0, y_0) = \frac{1}{2}\frac{\partial^2 C_Y^{(k-1)}(t, y \mid t_0, y_0)}{\partial y^2} - \frac{\partial C_Y^{(k-1)}(t, y \mid t_0, y_0)}{\partial t} + \frac{1}{2}\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial C_Y^{(h)}(t, y \mid t_0, y_0)}{\partial y}\frac{\partial C_Y^{(k-1-h)}(t, y \mid t_0, y_0)}{\partial y} \tag{12}$$

The binomial coefficient is written as $\binom{k}{j} \equiv \frac{k!}{j!(k-j)!}$ in the paper. This way of expanding log-transition function allows us to calculate the log-likelihood more easily. Taking exponential of the log-likelihood expansion yields an approximate transition density function that is positive. In Section 3.2.1, we will prove these results in a more general setting of an $m$dimensional diffusion case. In fact, Kolmogorov equations play a key role in finding the coefficients of (9) and (11). So, let us call this Kolmogorov-equation method.

It is obvious that Kolmogorov-equation method cannot be applied to any of univariate diffusions regardless of whether it is time-homogeneous or time-inhomogeneous when the integration (5) cannot be done explicitly, for example $\sigma(t, x) = \left(\theta_{-1}x^{-1} + \theta_0 + \theta_1 x + \theta_2 x^{\theta_3}\right)e^{\theta_4 t}$, where θ_i's are parameters. However, we can still derive the approximate density function for time-homogeneous univariate diffusions if we use the Hermite-expansion method. This is not possible in the case of univariate time-inhomogeneous diffusion unless there is an explicit forms of $\partial^k \gamma(t, x)/\partial t^k$, $k \geq 0$, for instance, again $\sigma(t, x) = \left(\theta_{-1}x^{-1} + \theta_0 + \theta_1 x + \theta_2 x^{\theta_3}\right)e^{\theta_4 t}$. The coefficients, $\eta_Z^{(j)}(\Delta, y_0)$'s in the Hermite-expansion of $p_Y^{(J,K)}(\Delta, y|y_0)$ for a univariate time-homogeneous diffusion are approximated as a function of $\mu_Y(y)$ and its differentiations $d^k \mu_Y(y)/dy^k\big|_{y=y_0}$, $k \geq 1$ evaluated at $y = y_0$, where $\mu_Y(y) = \frac{\mu[\gamma^{inv}(y)]}{\sigma[\gamma^{inv}(y)]} - \frac{1}{2}\frac{\partial \sigma(x)}{\partial x}\Big|_{x=\gamma^{-1}(y)}$ (Aït-Sahalia (2002)). The approximate transition density function of the diffusion X_t needs to be recovered by the change of variable:

$$p_X^{(J,K)}(\Delta, x|x_0) = \frac{p_Y^{(J,K)}[\Delta, \gamma(x)|\gamma(x_0)]}{\sigma(x)}.$$

Because $\frac{\mu(w)}{\sigma(w)} - \frac{1}{2}\frac{\partial \sigma(w)}{\partial w}$ is a known function and $d\gamma^{inv}(y)/dy = 1\Big/\left[d\gamma(x)/dx|_{x=\gamma^{inv}(y)}\right] = \sigma\left(\gamma^{inv}(y)\right)$, $\eta_Z^{(j)}(\Delta, \gamma(x_0))$'s of $p_X^{(J,K)}(\Delta, x|x_0)$ depend on μ, σ and their derivatives at $x = x_0$. Therefore $p_X^{(J,K)}(\Delta, x|x_0)$ is obtainable even though it is impossible to find an antiderivative for $\gamma(x)$. Similarly to $\eta_Z^{(j)}(\Delta, y_0)$'s, the coefficients, $\eta_Z^{(j)}(t, t_0, y_0)$'s of $p_Y^{(J,K)}(t, y|t_0, y_0)$ are dependent on $\mu_Y(t, y) = \frac{\mu[\gamma^{inv}(t,y),t]}{\sigma[\gamma^{inv}(t,y),t]} - \frac{1}{2}\frac{\partial \sigma[\gamma^{inv}(t,y),t]}{\partial x} + \frac{\partial \gamma[\gamma^{inv}(t,y),t]}{\partial t}$ and its differentials with respect to t and y at $t = t_0$ and $y = y_0$. Unlike $p_X^{(J,K)}(\Delta, x|x_0)$, $p_X^{(J,K)}(t, x|t_0, x_0)$ cannot be found without knowing the explicit formula of $\partial \gamma(t, x)/\partial t$ since we must be able to compute

$$\frac{\partial \gamma^{inv}(t, y)}{\partial t}\bigg|_{t=t_0, y=y_0} = -\frac{\partial \gamma(t, x)/\partial t}{\sigma\left(\gamma^{inv}(t, y), t\right)}\bigg|_{t=t_0, y=\gamma(t_0, x_0)},$$

and higher order derivatives of $\gamma(t, x)$ with respect to t to get $\eta_Z^{(j)}(t, t_0, \gamma(t_0, x_0))$. As can be seen below, the same problem arises in the multivariate case. Fortunately, this problem is resolved if we use the irreducible method discussed in section 3.2.2.

3.2 Multivariate Time-Inhomogeneous Diffusions

First, we need to introduce reducibility of a diffusion process. If a multivariate diffusion process is reducible, we can use the Hermite-expansion or Kolmogorov-equation methods as we have done for a univariate process to find its log-likelihood expansion. On the other hand, if a multivariate diffusion is not reducible (or irreducible) we can no longer turn to any of these approaches. As we will see in section 3.2.2, the irreducible method enables us to find the approximate log-likelihood function of X_t without transforming it to unit diffusion.

Definition :*(Reducibility) The diffusion process X_t is said to be reducible to unit diffusion (or reducible, in short) if and only if there exists a one-to-one transformation of the diffusion process X_t into a diffusion process Y_t whose dispersion matrix σ_Y is the identity matrix. That is, there exists an invertible function $\gamma(t,x)$, infinitely differentiable in t and x on $[0,\infty)\times S_X$ such that $Y_t \equiv \gamma(t,X_t)$ satisfies the stochastic differential equation*

$$dY_t = \mu_Y(t,Y_t)\,dt + dW_t$$

on the domain S_Y. The drift function of the i-th component of Y_t is

$$\begin{aligned}\mu_{Y_i}(t,y) \quad = \quad & \frac{\partial \gamma_i\left(t,\gamma^{inv}(t,y)\right)}{\partial t} + \sum_{p=1}^{m} \mu_p\left(t,\gamma^{inv}(t,y)\right) \left.\frac{\partial \gamma_i(t,x)}{\partial x_p}\right|_{x=\gamma^{inv}(t,y)} \\ & + \frac{1}{2}\sum_{p=1}^{m}\sum_{q=1}^{m}\sum_{r=1}^{m} \sigma_{pr}\left(t,\gamma^{inv}(t,y)\right)\sigma_{qr}\left(t,\gamma^{inv}(t,y)\right) \left.\frac{\partial^2 \gamma_i(t,x)}{\partial x_p \partial x_q}\right|_{x=\gamma^{inv}(t,y)}\end{aligned}$$

and

$$\nabla\gamma(t,x) = \sigma^{-1}(t,x) \tag{13}$$

by applying the Ito's lemma.

In the above definition, $\nabla\gamma(t,x)$ represents the Jacobian matrix of $\gamma(t,x)$ with respect to $x \in R^m$ such that $\nabla\gamma(t,x) = [\partial\gamma_i(t,x)/\partial x_j]_{i=1,\cdots,m;j=1,\cdots,m}$. Every univariate diffusion process can be transformed into a unit diffusion by (5). But, there doesn't always exist the transformation $\gamma(t,x)$ for a multivariate diffusion. We can tell whether a diffusion process is reducible or not by checking the conditions given below.

Proposition 1 *(Necessary and sufficient condition for reducibility) The diffusion process X_t is reducible if and only if*

$$\sum_{l=1}^{m} \frac{\partial \sigma_{ik}(t,x)}{\partial x_l}\sigma_{lj}(t,x) = \sum_{l=1}^{m} \frac{\partial \sigma_{ij}(t,x)}{\partial x_l}\sigma_{lk}(t,x)$$

for each x in S_X and triplet $\{i,j,k\} \subset \{1,2,\cdots,m\}$ such that $k > j$. If σ is nonsingular, then the condition can be expressed as

$$\frac{\partial \sigma_{ij}^{-1}(t,x)}{\partial x_k} = \frac{\partial \sigma_{ik}^{-1}(t,x)}{\partial x_j},$$

where $\sigma_{ij}^{-1}(t,x)$ is the (i,j) element of the inverse matrix of $\sigma(t,x)$.

Reducibility conditions have to do with only the dispersion matrix $\sigma(t,x)$. According to this proposition, when σ *is nonsingular,* $m^2(m-1)/2$ equalities must hold in order for an m-dimensional diffusion to be reducible. For example, when $m=2$ we need to check only two equalities.

$$\frac{\partial\sigma_{11}^{-1}(t,x)}{\partial x_2}=\frac{\partial\sigma_{12}^{-1}(t,x)}{\partial x_1}\text{ and }\frac{\partial\sigma_{22}^{-1}(t,x)}{\partial x_1}=\frac{\partial\sigma_{21}^{-1}(t,x)}{\partial x_2} \tag{14}$$

If $m=3$, in addition to (14), we need to look into seven more equalities, which are

$$\frac{\partial\sigma_{11}^{-1}(t,x)}{\partial x_3}=\frac{\partial\sigma_{13}^{-1}(t,x)}{\partial x_1},\frac{\partial\sigma_{22}^{-1}(t,x)}{\partial x_3}=\frac{\partial\sigma_{23}^{-1}(t,x)}{\partial x_2},\frac{\partial\sigma_{33}^{-1}(t,x)}{\partial x_1}=\frac{\partial\sigma_{31}^{-1}(t,x)}{\partial x_3},$$
$$\frac{\partial\sigma_{33}^{-1}(t,x)}{\partial x_2}=\frac{\partial\sigma_{32}^{-1}(t,x)}{\partial x_3},\frac{\partial\sigma_{21}^{-1}(t,x)}{\partial x_3}=\frac{\partial\sigma_{23}^{-1}(t,x)}{\partial x_1},$$
$$\frac{\partial\sigma_{12}^{-1}(t,x)}{\partial x_3}=\frac{\partial\sigma_{13}^{-1}(t,x)}{\partial x_2}\text{ and }\frac{\partial\sigma_{31}^{-1}(t,x)}{\partial x_2}=\frac{\partial\sigma_{32}^{-1}(t,x)}{\partial x_1}$$

3.2.1 Reducible Diffusions

To construct approximate log-transition density function of a multivariate time-inhomogeneous reducible diffusion process we can follow the same line as the univariate case. Let us first discuss Hermite-expansion approach. Multivariate Hermite polynomials are defined by

$$H_h(z)=\frac{(-1)^{|h|}}{\phi(z)}\frac{\partial^{|h|}\phi(z)}{\partial z_1^{h_1}\cdots\partial z_m^{h_m}},$$

where h is an $m-$dimensional vector with non-negative integers h_i's and $\phi(z)$ is the density of the m-dimensional multivariate Normal distribution with mean zero and identity covariance matrix. In order to represent the sum of the all elements of an m dimensional vector h, $|h|$ will be used, i.e., $|h|=\sum_{i=1}^{m}h_i$. The Hermite polynomials can be computed explicitly to any order $h=(h_1,\cdots,h_m)$ (see Chapter 5 of McCullagh (1987) or Withers (2000)). The Hermite polynomials are orthogonal in the following sense

$$\int_{R^m}H_h(z)H_k(z)\phi(z)\,dx=\begin{cases}h_1!\cdots h_m! & \text{if } h=k\\ 0 & \text{otherwise}\end{cases}.$$

The J-th order Hermite series expansion, $p_Y^{(J)}$ for a multivariate time-inhomogeneous unit diffusion process Y_t is of the form

$$p_Y^{(J)}(t,y\mid t_0,y_0)=\Delta^{-m/2}\phi\left(\frac{y-y_0}{\sqrt{\Delta}}\right)\sum_{h\in N^m:|h|\le J}\eta_h(t,t_0,y_0)H_h\left(\frac{y-y_0}{\sqrt{\Delta}}\right)$$

Using the orthogonality of the Hermite polynomials,

$$\eta_h(t,t_0,y_0)=\frac{1}{h_1!\cdots h_m!}E\left[H_h\left(\frac{Y_t-y_0}{\sqrt{\Delta}}\right)\middle|Y_{t_0}=y_0\right].$$

If the coefficient, $\eta_h(t, t_0, y_0)$ is approximated up to K-th order by using the infinitesimal generator A_Y corresponding to the diffusion process Y_t, i.e.,

$$A_Y \circ f(t, y, t_0, y_0) = \frac{\partial f(t, y, t_0, y_0)}{\partial t} + \sum_{i=1}^{m} \mu_{Y_i}(t, y) \frac{\partial f(t, y, t_0, y_0)}{\partial y_i} + \frac{1}{2} \sum_{i=1}^{m} \sum_{j=1}^{m} \frac{\partial^2 f(t, y, t_0, y_0)}{\partial y_i \partial y_j}.$$

then, we attain

$$p_Y^{(J,K)}(t, y|t_0, y_0) = \Delta^{-m/2} \phi\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \left\{ \sum_{h \in N^m : |h| \leq J} \frac{1}{h_1! \cdots h_m!} \left[\sum_{i=0}^{K} \frac{\Delta^i}{i!} A_Y^i \circ H_h\left(\frac{y - y_0}{\sqrt{\Delta}}\right)\bigg|_{y=y_0} \right] H_h\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \right\}.$$

We can send J to ∞ and rearrange the terms of $p_Y^{(J,K)}$ in ascending orders of Δ to acquire an alternative expansion $p_Y^{(K)} = p_Y^{(\infty,K)}$ such that

$$p_Y^{(K)}(t, y|t_0, y_0) = \Delta^{-m/2} \phi\left(\frac{y - y_0}{\sqrt{\Delta}}\right) \exp\left[\sum_{i=1}^{m} (y_i - y_{0i}) \int_0^1 \mu_{Y_i}(t, y_0 + u(y - y_0)) \, du \right] \sum_{k=0}^{K} c_Y^{(k)}(t, y|t_0, y_0) \frac{\Delta^k}{k!}. \tag{15}$$

Taking logarithm of (15) and Taylor-expanding it in Δ around zero we get the approximate log-transition density of Y_t in Theorem 1 as a result of applying the Kolmogorov-equation method.

Theorem 1 *The K-th order log-transition density expansion of a time-inhomogeneous multivariate unit diffusion process Y_t is*

$$l_Y^{(K)}(t, y \mid t_0, y_0) = -\frac{m}{2} \ln(2\pi\Delta) + \frac{C_Y^{(-1)}(t, y \mid t_0, y_0)}{\Delta} + \sum_{k=0}^{K} C_Y^{(k)}(t, y \mid t_0, y_0) \frac{\Delta^k}{k!} \tag{16}$$

with

$$C_Y^{(-1)}(t, y \mid t_0, y_0) = -\frac{1}{2} \sum_{i=1}^{m} (y_i - y_{0i})^2 \tag{17}$$

$$C_Y^{(0)}(t, y \mid t_0, y_0) = \sum_{i=1}^{m} (y_i - y_{0i}) \int_0^1 \mu_{Y_i}(t, y_0 + u(y - y_0)) \, du \tag{18}$$

and for $k \geq 1$

$$C_Y^{(k)}(t, y \mid t_0, y_0) = k \int_0^1 G_Y^{(k)}(t, y_0 + u(y - y_0) \mid t_0, y_0) \, u^{k-1} du \tag{19}$$

where

$$\begin{aligned} G_Y^{(1)}(t, y \mid t_0, y_0) \quad = \quad & -\sum_{i=1}^{m} \frac{\partial \mu_{Y_i}(t, y)}{\partial y_i} - \frac{\partial C_Y^{(0)}(t, y \mid t_0, y_0)}{\partial t} \\ & -\sum_{i=1}^{m} \mu_{Y_i}(t, y) \frac{\partial C_Y^{(0)}(t, y \mid t_0, y_0)}{\partial y_i} \\ & +\frac{1}{2} \sum_{i=1}^{m} \left\{ \frac{\partial^2 C_Y^{(0)}(t, y \mid t_0, y_0)}{\partial y_i^2} + \left[\frac{\partial C_Y^{(0)}(t, y \mid t_0, y_0)}{\partial y_i} \right]^2 \right\} \end{aligned} \tag{20}$$

and for $k \geq 2$

$$G_Y^{(k)}(t, y \mid t_0, y_0) = \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2 C_Y^{(k-1)}(t, y \mid t_0, y_0)}{\partial y_i^2} - \frac{\partial C_Y^{(k-1)}(t, y \mid t_0, y_0)}{\partial t} - \sum_{i=1}^{m} \mu_{Y_i}(t, y) \frac{\partial C_Y^{(k-1)}(t, y \mid t_0, y_0)}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{m}\sum_{h=0}^{k-1} \binom{k-1}{h} \frac{\partial C_Y^{(h)}(t, y \mid t_0, y_0)}{\partial y_i} \frac{\partial C_Y^{(k-1-h)}(t, y \mid t_0, y_0)}{\partial y_i}. \tag{21}$$

It is well known that the transition density $p_Y(t, y|t_0, y_0)$ satisfies the Kolmogorov forward and backward equations, respectively,

$$\frac{\partial p_Y(t, y \mid t_0, y_0)}{\partial t} = -\sum_{i=1}^{m} \frac{\partial \left\{\mu_{Y_i}(t, y) p_Y(t, y \mid t_0, y_0)\right\}}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2 p_Y(t, y \mid t_0, y_0)}{\partial y_i^2}$$

and

$$-\frac{\partial p_Y(t, y \mid t_0, y_0)}{\partial t_0} = \sum_{i=1}^{m} \mu_{Y_i}(t_0, y_0) \frac{\partial p_Y(t, y \mid t_0, y_0)}{\partial y_{0i}} + \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2 p_Y(t, y \mid t_0, y_0)}{\partial y_{0i}^2}.$$

If we plug the log-likelihood expansion (16) into the Kolmogorov forward equation for $l_Y(t, y \mid t_0, y_0)$:

$$\frac{\partial l_Y(t, y \mid t_0, y_0)}{\partial t} = -\sum_{i=1}^{m} \frac{\partial \mu_{Y_i}(t, y)}{\partial y_i} - \sum_{i=1}^{m} \mu_{Y_i}(t, y) \frac{\partial l_Y(t, y \mid t_0, y_0)}{\partial y_i} + \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2 l_Y(t, y \mid t_0, y_0)}{\partial y_i^2} + \frac{1}{2}\sum_{i=1}^{m} \left[\frac{\partial l_Y(t, y \mid t_0, y_0)}{\partial y_i}\right]^2 \tag{22}$$

and equate the coefficients of $\Delta^k,\ k \geq -2$ on both sides of (22) then we get the PDEs for $C_Y^{(k)}(t, y \mid t_0, y_0)\,,\ k \geq -1$. Each of these PDEs can be solved as given in Theorem 1 by using Gaussianity of Y_t as $\Delta \to 0$, boundary conditions and the fact that for each term $\Delta^k,\ k \geq 2$, (16) also satisfies the Kolmogorov backward equation for $l_Y(t, y \mid t_0, y_0)$:

$$-\frac{\partial l_Y(t, y \mid t_0, y_0)}{\partial t_0} = \sum_{i=1}^{m} \mu_{Y_i}(t_0, y_0) \frac{\partial l_Y(t, y \mid t_0, y_0)}{\partial y_{0i}} + \frac{1}{2}\sum_{i=1}^{m} \frac{\partial^2 l_Y(t, y \mid t_0, y_0)}{\partial y_{0i}^2} + \frac{1}{2}\sum_{i=1}^{m} \left[\frac{\partial l_Y(t, y \mid t_0, y_0)}{\partial y_{0i}}\right]^2. \tag{23}$$

The log-density, $l_X(t, x \mid t_0, x_0)$ of the original process X_t can be retrieved from $l_Y(t, y \mid t_0, y_0)$ by change of variable as

$$\begin{aligned} l_X(t, x \mid t_0, x_0) &= \ln\left(Det\left[\nabla\gamma(t, x)\right]\right) + l_Y(t, \gamma(t, x) \mid t_0, \gamma(t_0, x_0)) \\ &= -D_\upsilon(t, x) + l_Y(t, \gamma(t, x) \mid t_0, \gamma(t_0, x_0)). \end{aligned}$$

The second equality is because $Det\left[\nabla\gamma(t, x)\right] = Det\left[\sigma^{-1}(t, x)\right] = Det\left[\upsilon(t, x)\right]^{-1/2}$ and we define $D_\upsilon(t, x) \equiv \frac{1}{2}\ln\left(Det\left[\upsilon(t, x)\right]\right)$. The determinant of a matrix a will be written as $Det\,[a]$. Replace l_Y with $l_Y^{(K)}$ found in Theorem 1 and define $l_X^{(K)}$ as

$$
\begin{aligned}
l_X^{(K)}\left(t,x\mid t_0,x_0\right) &= -D_{\upsilon}\left(t,x\right)+l_Y^{(K)}\left(t,\gamma\left(t,x\right)\mid t_0,\gamma\left(t_0,x_0\right)\right) \\
&= -D_{\upsilon}\left(t,x\right)-\frac{m}{2}\ln\left(2\pi\Delta\right)+\frac{C_Y^{(-1)}\left(t,\gamma\left(t,x\right)\mid t_0,\gamma\left(t_0,x_0\right)\right)}{\Delta} \\
&\quad +\sum_{k=0}^{K}C_Y^{(k)}\left(t,\gamma\left(t,x\right)\mid t_0,\gamma\left(t_0,x_0\right)\right)\frac{\Delta^k}{k!}.
\end{aligned}
\tag{24}
$$

By construction, $l_X^{(K)}$ solves the Kolmogorov equations for X_t at the same order as $l_Y^{(K)}$.

If $\mu_i\left(t,x\right)=\mu_i\left(t,x_i\right)$ and $\sigma_{ii}\left(t,x\right)=\sigma_{ii}\left(t,x_i\right)$ for all $i=1,\cdots,m$ and $\sigma_{ij}\left(t,x_i\right)=0$ for $i\neq j$ then the transition density of the diffusion is just the product of the transition density of each variable. Reducibility of this special case is obvious and

$$
l_X^{(K)}\left(t,x\mid t_0,x_0\right)=\sum_{i=1}^{m}l_{X_i}^{(K)}\left(t,x_i\mid t_0,x_{0i}\right)
$$

where $l_{X_i}^{(K)}\left(t,x_i\mid t_0,x_{0i}\right)$ is the univariate expansion of the log-density for the i-th variable.

3.2.2 Irreducible Diffusions

As mentioned earlier, not every multivariate time-inhomogeneous diffusion is reducible. Moreover, the reducibility conditions in Proposition 1 are so restrictive that many of multivariate time-inhomogeneous diffusions are likely to be irreducible. If a multivariate time-inhomogeneous diffusion process, X_t is irreducible, none of Hermite-expansion and Kolmogorov-equation methods can be adopted since we cannot transform it into a unit diffusion process. These approaches, in fact, yield different ways of expanding the identical approximate log-density function of a reducible diffusion. Furthermore, as discussed in the univariate case, even if X_t is time-homogeneous and reducible, if there is no explicit formula for γ_i^{inv}, only the Hermite-expansion method is available. On the other hand, if it is time-inhomogeneous and there doesn't exist $\partial^k\gamma_i/\partial t^k,\ k\geq 1$, in closed form either, neither of the methods can be applied but we can turn to the irreducible method explained below.

The key idea of finding approximate log-transition density of irreducible diffusions is to postulate the form of the log-likelihood expansion of X_t as the one found from the reducible case:

$$
l_X^{(K)}\left(t,x\mid t_0,x_0\right)=-\frac{m}{2}\ln\left(2\pi\Delta\right)-D_{\upsilon}\left(t,x\right)+\frac{C_X^{(-1)}\left(t,x\mid t_0,x_0\right)}{\Delta}+\sum_{k=0}^{K}C_X^{(k)}\left(t,x\mid t_0,x_0\right)\frac{\Delta^k}{k!}
\tag{25}
$$

Replacing l_X with $l_X^{(K)}$ in the Kolmogorov equations for the log-transition density of X_t and matching the terms with the same orders of Δ yields PDEs of the coefficients $C_X^{(k)}\left(t,x\mid t_0,x_0\right),\ k\geq -1$ as given in

Theorem 2. The Kolmogorov forward and backward equations for $l_X(t,x \mid t_0,x_0)$ are, respectively,

$$\begin{aligned}\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial t} &= -\sum_{i=1}^{m}\frac{\partial \mu_i(t,x)}{\partial x_i}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2 \upsilon_{ij}(t,x)}{\partial x_i \partial x_j}-\sum_{i=1}^{m}\mu_i(t,x)\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial x_i}\\ &+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial \upsilon_{ij}(t,x)}{\partial x_i}\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial x_j}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial^2 l_X(t,x\mid t_0,x_0)}{\partial x_i\partial x_j} \qquad (26)\\ &+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial x_i}\upsilon_{ij}(t,x)\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial x_j}\end{aligned}$$

and

$$\begin{aligned}-\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial t_0} &= \sum_{i=1}^{m}\mu_i(t_0,x_0)\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial x_{0i}}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t_0,x_0)\frac{\partial^2 l_X(t,x\mid t_0,x_0)}{\partial x_{0i}\partial x_{0j}} \qquad (27)\\ &+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial x_{0i}}\upsilon_{ij}(t_0,x_0)\frac{\partial l_X(t,x\mid t_0,x_0)}{\partial x_{0j}}.\end{aligned}$$

Theorem 2 *The coefficients of the log-likelihood expansions of irreducible multivariate time-inhomogeneous diffusion processes, $C_X^{(k)}(t,x|t_0,x_0)$'s in (25) satisfy the following differential equations*

$$f_X^{(k-1)}(t,x\mid t_0,x_0)=0$$

where $k=-1,0,\cdots,K$,

$$f_X^{(-2)}(t,x\mid t_0,x_0)=-2C_X^{(-1)}(t,x\mid t_0,x_0)-\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_i}\frac{\partial C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_j}, \qquad (28)$$

$$f_X^{(-1)}(t,x\mid t_0,x_0)=-\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_i}\frac{\partial C_X^{(0)}(t,x\mid t_0,x_0)}{\partial x_j}-G_X^{(0)}(t,x\mid t_0,x_0), \qquad (29)$$

and for $k\geq 1$

$$\begin{aligned}f_X^{(k-1)}(t,x\mid t_0,x_0) &= C_X^{(k)}(t,x\mid t_0,x_0) \qquad (30)\\ &-\frac{1}{k}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_i}\frac{\partial C_X^{(k)}(t,x\mid t_0,x_0)}{\partial x_j}\\ &-G_X^{(k)}(t,x\mid t_0,x_0).\end{aligned}$$

For $k=0$ *and* 1,

$$G_X^{(0)}(t,x\mid t_0,x_0)=-\frac{\partial C_X^{(-1)}(t,x|t_0,x_0)}{\partial t}+G_X^{(0,1)}(t,x\mid t_0,x_0)+G_X^{(0,3)}(t,x\mid t_0,x_0),$$

where

$$
\begin{aligned}
G_X^{(0,1)}(t,x \mid t_0,x_0) &= -\sum_{i=1}^{m}\mu_i(t,x)\frac{\partial C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_i} \\
&+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial \upsilon_{ij}(t,x)}{\partial x_i}\frac{\partial C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_j} \\
&-\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_i}\frac{\partial D_\upsilon(t,x)}{\partial x_j} \\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial^2 C_X^{(-1)}(t,x\mid t_0,x_0)}{\partial x_i\partial x_j}
\end{aligned}
$$

and

$$
G_X^{(0,3)}(t,x\mid t_0,x_0)=\frac{m}{2},
$$

and

$$
G_X^{(1)}(t,x\mid t_0,x_0)=-\frac{\partial C_X^{(0)}(t,x|t_0,x_0)}{\partial t}+G_X^{(1,1)}(t,x\mid t_0,x_0)+G_X^{(1,2)}(t,x\mid t_0,x_0)+G_X^{(1,3)}(t,x\mid t_0,x_0),
$$

where

$$
\begin{aligned}
G_X^{(1,1)}(t,x \mid t_0,x_0) &= -\sum_{i=1}^{m}\mu_i(t,x)\frac{\partial C_X^{(0)}(t,x\mid t_0,x_0)}{\partial x_i} \\
&+\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial \upsilon_{ij}(t,x)}{\partial x_i}\frac{\partial C_X^{(0)}(t,x\mid t_0,x_0)}{\partial x_j} \\
&-\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial C_X^{(0)}(t,x\mid t_0,x_0)}{\partial x_i}\frac{\partial D_\upsilon(t,x)}{\partial x_j} \\
&+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial^2 C_X^{(0)}(t,x\mid t_0,x_0)}{\partial x_i\partial x_j},
\end{aligned}
$$

$$
G_X^{(1,2)}(t,x\mid t_0,x_0)=\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\frac{\partial C_X^{(0)}(t,x\mid t_0,x_0)}{\partial x_i}\frac{\partial C_X^{(0)}(t,x\mid t_0,x_0)}{\partial x_j},
$$

and

$$
\begin{aligned}
G_X^{(1,3)}(t,x\mid t_0,x_0) &= \frac{\partial D_\upsilon(t,x)}{\partial t}-\sum_{i=1}^{m}\frac{\partial\mu_i(t,x)}{\partial x_i}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^2\upsilon_{ij}(t,x)}{\partial x_i\partial x_j}+\sum_{i=1}^{m}\mu_i(t,x)\frac{\partial D_\upsilon(t,x)}{\partial x_i} \\
&-\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial\upsilon_{ij}(t,x)}{\partial x_i}\frac{\partial D_\upsilon(t,x)}{\partial x_j} \\
&-\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t,x)\left[\frac{\partial^2 D_\upsilon(t,x)}{\partial x_i\partial x_j}-\frac{\partial D_\upsilon(t,x)}{\partial x_i}\frac{\partial D_\upsilon(t,x)}{\partial x_j}\right].
\end{aligned}
$$

For $k\geq 2$

$$
G_X^{(k)}(t,x\mid t_0,x_0)=-\frac{\partial C_X^{(k-1)}(t,x|t_0,x_0)}{\partial t}+G_X^{(k,1)}(t,x\mid t_0,x_0)+G_X^{(k,2)}(t,x\mid t_0,x_0),
$$

where

$$\begin{aligned} G_X^{(k,1)}(t,x \mid t_0,x_0) &= -\sum_{i=1}^m \mu_i(t,x) \frac{\partial C_X^{(k-1)}(t,x \mid t_0,x_0)}{\partial x_i} \\ &\quad + \sum_{i=1}^m \sum_{j=1}^m \frac{\partial \upsilon_{ij}(t,x)}{\partial x_i} \frac{\partial C_X^{(k-1)}(t,x \mid t_0,x_0)}{\partial x_j} \\ &\quad - \sum_{i=1}^m \sum_{j=1}^m \upsilon_{ij}(t,x) \frac{\partial C_X^{(k-1)}(t,x \mid t_0,x_0)}{\partial x_i} \frac{\partial D_\upsilon(t,x)}{\partial x_j} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \upsilon_{ij}(t,x) \frac{\partial^2 C_X^{(k-1)}(t,x \mid t_0,x_0)}{\partial x_i \partial x_j} \end{aligned}$$

and

$$\begin{aligned} G_X^{(k,2)}(t,x \mid t_0,x_0) &= \sum_{i=1}^m \sum_{j=1}^m \upsilon_{ij}(t,x) \frac{\partial C_X^{(0)}(t,x \mid t_0,x_0)}{\partial x_i} \frac{\partial C_X^{(k-1)}(t,x \mid t_0,x_0)}{\partial x_j} \\ &\quad + \frac{1}{2} \sum_{i=1}^m \sum_{j=1}^m \upsilon_{ij}(t,x) \left[\sum_{h=1}^{k-2} \binom{k-1}{h} \frac{\partial C_X^{(h)}(t,x \mid t_0,x_0)}{\partial x_i} \frac{\partial C_X^{(k-1-h)}(t,x \mid t_0,x_0)}{\partial x_j} \right]. \end{aligned}$$

Unlike the reducible case, the PDEs in Theorem 2 are in general unsolvable analytically. But we can Taylor-expand each coefficient, $C_X^{(k)}(t,x \mid t_0,x_0)$ around (t_0,x_0) up to j_k-th order in order to achieve the same approximation error of $O_p\left(\Delta^{K+1}\right)$ for each coefficient. We denote it as $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$ where j_k is the maximum order of the Taylor approximation. So, $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$ can be written as:

$$C_X^{(j_k,k)}(t,x \mid t_0,x_0) = \sum_{0 \le a_0+a_1+\cdots+a_m \le j_k} c^{(k)}_{0_{a_0}1_{a_1}\cdots m_{a_m}} \Delta^{a_0} (x_1-x_{01})^{a_1} (x_2-x_{02})^{a_2} \cdots (x_m-x_{0m})^{a_m}, \quad (31)$$

where a_q's are non-negative integers, $q = 0, 1, \cdots, m$ and we use $c^{(k)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$ instead of $c^{(k)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}(t_0,x_0)$ to simplify expressions. If $a_q = 0$ we omit q-th index, for instance, if $a_q = 0$ for all $0 \le q \le m$, $c^{(k)}_{0_{a_0}1_{a_1}\cdots m_{a_m}} = c^{(k)}$, which is the constant term of $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$. When $a_q = 1$ we write just subscript q rather than q_1.

Superseding the coefficients $C_X^{(k)}(t,x \mid t_0,x_0)$'s of (25) with $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$ gives the following K-th order log-density function $\widetilde{l}_X^{(K)}(\Delta,t,x \mid t_0,x_0)$.

$$\widetilde{l}_X^{(K)}(t,x \mid t_0,x_0) = -\frac{m}{2}\ln(2\pi\Delta) - D_\upsilon(t,x) + \frac{C_X^{(j_{-1},-1)}(t,x \mid t_0,x_0)}{\Delta} + \sum_{k=0}^{K} C_X^{(j_k,k)}(t,x \mid t_0,x_0)\frac{\Delta^k}{k!} \quad (32)$$

and

$$\left| l_X^{(K)}(t,x \mid t_0,x_0) - \widetilde{l}_X^{(K)}(t,x \mid t_0,x_0) \right| = O_p\left(\Delta^{K+1}\right).$$

It is important to not make $D_\upsilon(t,x)$, which is known, part of $C_X^{(0)}(t,x \mid t_0,x_0)$ and to not Taylor-expand it so that $\widetilde{l}_X^{(K)}(t,x \mid t_0,x_0)$ can also solve the backward equation (27). Now, we just need to find the appropriate order j_k for each coefficient $C_X^{(k)}(t,x \mid t_0,x_0)$ and the coefficients of the Taylor-expansion $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$, which are to be discussed below. Let us call what we have discussed the irreducible method.

For each k, j_k is determined to make the approximation error of each term, $C_X^{(j_k,k)}(t,x \mid t_0,x_0)\,\Delta^k$ same as that of (25), $O_p\left(\Delta^{K+1}\right)$. Because $x-x_0=O_p\left(\Delta^{1/2}\right)$,

$$\left|C_X^{(k)}(t,x \mid t_0,x_0)\,\Delta^k - C_X^{(j_k,k)}(t,x \mid t_0,x_0)\,\Delta^k\right| = O_p\left((x-x_0)^{j_k-a_0}\,\Delta^{a_0}\Delta^k\right) = O_p\left(\Delta^{(j_k-a_0)/2+a_0+k}\right).$$

Therefore, we have to set $j_k=2\,(K+1-k)-a_0$, which can be obtained by solving $(j_k-a_0)/2+a_0+k=K+1$. Notice that the maximum order of $(x-x_0)$, j_k-a_0 decreases twice as fast as that of Δ, as a_0 increases. Considering the terms without Δ in the expansion, i.e. $a_0=0$, $j_k=2\,(K-k)$, which is of course the same as the order for time-homogenous diffusion process. However, having Δ term in the Taylor-expansion of the coefficients makes the maximum order of the terms with Δ less than that of the terms without Δ. For each k, the combinations of $(a_0,|a|)$, where $a=(a_1,\cdots,a_m)$, to calculate $\widetilde{l}_X^{(K)}$ are as follows:

$k=-1$	$(a_0,\lvert a\rvert)\in\{(0,2\,(K+2))\,,(1,2\,(K+1))\,,(2,2K)\,,\cdots,((K+1)\,,2)\,,((K+2)\,,0)\}$	
$k=0$	$(a_0,\lvert a\rvert)\in\{(0,2\,(K+1))\,,(1,2K)\,,(2,2\,(K-1))\,,\cdots,(K,2)\,,((K+1)\,,0)\}$	
$k=1$	$(a_0,\lvert a\rvert)\in\{(0,2K)\,,(1,2\,(K-1))\,,(2,2\,(K-2))\,,\cdots,((K-1)\,,2)\,,(K,0)\}$	
$\vdots$	$\vdots$	(33)
$k=K-1$	$(a_0,\lvert a\rvert)\in\{(0,4)\,,(1,2)\,,(2,0)\}$	
$k=K$	$(a_0,\lvert a\rvert)\in\{(0,2)\,,(1,0)\}$	
$k=K+1$	$(a_0,\lvert a\rvert)\in\{(0,0)\}$	

The reason why we need to include $C_X^{(0,K+1)}$, even though we compute $\widetilde{l}_X^{(K)}$, is explained below.

In order to find the Taylor-approximation coefficients of $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$, we first need to put $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$ and the j_k-th order Taylor-expansions of $\mu_i(t,x)$, $\upsilon_{ij}(t,x)$ and $D_\upsilon(t,x)$, respectively, in places of $C_X^{(k)}(t,x \mid t_0,x_0)$, $\mu_i(t,x)$, $\upsilon_{ij}(t,x)$ and $D_\upsilon(t,x)$ in $f_X^{(k-1)}(t,x|t_0,x_0)$. Analogously to (32), $\widetilde{f}_X^{(k-1)}(t,x|t_0,x_0)$, $k\geq 0$ denote the corresponding $f_X^{(k-1)}(t,x|t_0,x_0)$ after this replacement. Notice that $G_X^{(k)}(t,x \mid t_0,x_0)$ depends on $\mu_i(t,x)$, $\upsilon_{ij}(t,x)$, and all the former coefficients, $C_X^{(h)}(t,x \mid t_0,x_0)$, $h\leq k-1$.

In the time-homogeneous case, $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$'s and $f_X^{(k-1)}(t,x|t_0,x_0)$'s are independent of time variable. Equating the coefficients of the same orders in $\widetilde{f}_X^{(-2)}(x|x_0)=0$ to zeros and solving the simultaneous equations of the coefficients enable us to attain $C_X^{(-1,j_{-1})}(x \mid x_0)$. These coefficients must be calculated recursively from low to high order because the coefficients of higher order terms depend on those of lower order terms. Likewise, given $C_X^{(j_{-1},-1)}(x \mid x_0)$, the coefficients of $C_X^{(j_0,0)}(x \mid x_0)$ need to be computed from low to high order. Note that the differential equation $f_X^{(k-1)}(x|x_0)$ for $C_X^{(k)}(x \mid x_0)$ is dependent on $C_X^{(h)}(x \mid x_0)$, $h\leq k-1$. Therefore, each of the subsequent Taylor-expansions, $C_X^{(j_k,k)}(x \mid x_0)$'s can be acquired in the same way using $C_X^{(j_h,h)}(x \mid x_0)$, $h\leq k-1$, and the coefficients of $\widetilde{l}_X^{(K)}(x \mid x_0)$ have to be found recursively in increasing order of Δ.

Proposition 2 *Suppose that the diffusion X is reducible and let $l_X^{(K)}$ be its approximate log-transition density function obtained by using the reducible method. If we calculate its log-likelihood expansion, $\widetilde{l}_X^{(K)}$ by using irreducible method. Then, each coefficient $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$ from $\widetilde{l}_X^{(K)}$ is a Taylor-expansion in (t,x) about (t_0,x_0) at order j_k of the coefficient $C_X^{(k)}(t,x \mid t_0,x_0) = C_X^{(k)}[t,\gamma(t,x) \mid t_0,\gamma(t_0,x_0)]$ from $l_X^{(K)}$.*

It is critical to obtain all coefficients of $C_X^{(j_k,k)}(t,x \mid t_0,x_0)$ at each step of solving simultaneous equations. However, this recursive way of finding the approximate coefficients breaks down if it is time-inhomogeneous, to be precise if σ is a function of t as well as x, since infinitely many coefficients of $C_X^{(j_{-1},-1)}(t,x \mid t_0,x_0)$ and $C_X^{(j_0,0)}(t,x \mid t_0,x_0)$ cannot be determined from $\widetilde{f}_X^{(-2)}(t,x \mid t_0,x_0) = 0$ and $\widetilde{f}_X^{(-1)}(t,x \mid t_0,x_0) = 0$, respectively. First, it is clear from $f_X^{(-1)}(t,x|t_0,x_0)$ that $c_{0_{a_0}}^{(0)}$, $a_0 \geq 1$ are unobtainable because $f_X^{(-1)}(t,x|t_0,x_0)$ involves only $\frac{\partial C_X^{(0)}(t,x|t_0,x_0)}{\partial x_j}$, $j = 1,\cdots,m$. Therefore we cannot find any of the coefficients of $(t-t_0)^{a_0}$, $c_{0_{a_0}}^{(0)}$, $a_0 \geq 1$, in $C_X^{(j_0,0)}(t,x \mid t_0,x_0)$ from $\widetilde{f}_X^{(-1)}(t,x|t_0,x_0) = 0$. Next, as is the case for time-homogeneous diffusion, the constant and first order terms of $C_X^{(j_{-1},-1)}(t,x \mid t_0,x_0)$ are zero. But, equating the second order terms of $\widetilde{f}_X^{(-2)}(t,x|t_0,x_0)$ to zeros,

$$
\begin{aligned}
& -2\begin{pmatrix}
c_{0_2}^{(-1)} & \frac{1}{2}c_{01}^{(-1)} & \frac{1}{2}c_{02}^{(-1)} & \cdots & \frac{1}{2}c_{0m}^{(-1)} \\
\frac{1}{2}c_{01}^{(-1)} & c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\
\frac{1}{2}c_{02}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\frac{1}{2}c_{0m}^{(-1)} & \frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)}
\end{pmatrix} \\
= \; & 4\begin{pmatrix}
\frac{1}{2}c_{01}^{(-1)} & \frac{1}{2}c_{02}^{(-1)} & \cdots & \frac{1}{2}c_{0m}^{(-1)} \\
c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\
\frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)}
\end{pmatrix} \upsilon(t_0,x_0) \begin{pmatrix}
\frac{1}{2}c_{01}^{(-1)} & \frac{1}{2}c_{02}^{(-1)} & \cdots & \frac{1}{2}c_{0m}^{(-1)} \\
c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\
\frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)}
\end{pmatrix}^T .
\end{aligned}
$$

Transposition of a matrix a will be written as a^T. From this

$$
\begin{pmatrix}
c_{1_2}^{(-1)} & \frac{1}{2}c_{12}^{(-1)} & \cdots & \frac{1}{2}c_{1m}^{(-1)} \\
\frac{1}{2}c_{12}^{(-1)} & c_{2_2}^{(-1)} & \cdots & \frac{1}{2}c_{2m}^{(-1)} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{2}c_{1m}^{(-1)} & \frac{1}{2}c_{2m}^{(-1)} & \cdots & c_{m_2}^{(-1)}
\end{pmatrix} = -\frac{1}{2}\upsilon(t_0,x_0)^{-1} \tag{34}
$$

and

$$
c_{0_2}^{(-1)} = -\frac{1}{2}\begin{pmatrix} c_{01}^{(-1)} \\ c_{02}^{(-1)} \\ \vdots \\ c_{0m}^{(-1)} \end{pmatrix}^T \upsilon(t_0,x_0) \begin{pmatrix} c_{01}^{(-1)} \\ c_{02}^{(-1)} \\ \vdots \\ c_{0m}^{(-1)} \end{pmatrix}.
$$

But $c_{01}^{(-1)}, \cdots, c_{0m}^{(-1)}$ are indeterminate and so is $c_{0_2}^{(-1)}$. Actually, the coefficients of $(t-t_0)^{a_0}(x_h - x_{0h})$, $c_{0_{a_0}h}^{(-1)}$, $h = 1, \cdots, m$ and $a_0 \geq 1$, cannot be determined from $\widetilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$. This can be shown from looking at the coefficients $c_{0_{a_0}h}^{(-1)}$ of $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$ in $\widetilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$ and using Lemma 1.

Lemma 1

$$\sum_{i=1}^{m} \upsilon_{ij}(t_0, x_0) \frac{\partial \widehat{C}_X^{(2,-1)}(t, x \mid t_0, x_0)}{\partial x_i} = -(x_j - x_{0j}),$$

where $\widehat{C}_X^{(2,-1)}(t, x \mid t_0, x_0)$ *is second order terms of the Taylor-expansion of* $C_X^{(-1)}(t, x \mid t_0, x_0)$ *around* (t_0, x_0) *except the terms* $(t - t_0)(x_i - x_{0i})$, $i = 1, \cdots, m$.

Consider $\widetilde{f}_X^{(-2)}(t, x \mid t_0, x_0) = 0$ to calculate the coefficients of $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$ such that

$$2C_X^{(j_{-1}-1)}(t, x \mid t_0, x_0) = -\sum_{i=1}^{m}\sum_{j=1}^{m} \upsilon_{ij}^{(j_{-1}-2)}(t, x) \frac{\partial C_X^{(j_{-1}-1)}(t, x \mid t_0, x_0)}{\partial x_i} \frac{\partial C_X^{(j_{-1}-1)}(t, x \mid t_0, x_0)}{\partial x_j}, \tag{35}$$

where $\upsilon_{ij}^{(j_{-1}-2)}(t, x)$ is Taylor-approximation of $\upsilon_{ij}(t, x)$ about (t_0, x_0) up to $(j_{-1} - 2)$-th order because the constant and first order terms of $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$ are zeros. Let us compute $c_{0_{a_0}h}^{(-1)}$ of $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$, $a_0 \geq 1$ and $h = 1, \cdots, m$. Because of the fact that

$$-\sum_{i=1}^{m} \upsilon_{ih}(t_0, x_0) \frac{\partial \widehat{C}_X^{(2,-1)}(t, x \mid t_0, x_0)}{\partial x_i} \frac{\partial c_{0_{a_0}h}^{(-1)}(t - t_0)^{a_0}(x_h - x_{0h})}{\partial x_h} = c_{0_{a_0}h}^{(-1)}(t - t_0)^{a_0}(x_h - x_{0h})$$

due to Lemma 1 and the symmetry of the right-hand side of equation (35), $c_{0_{a_0}h}^{(-1)}$ is cancelled out in (35). Thus, none of $c_{0_{a_0}h}^{(-1)}$, $a_0 \geq 1$ and $h = 1, \cdots, m$ in $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$ can be determined from $\widetilde{f}_X^{(-2)}(t, x \mid t_0, x_0) = 0$. Although all the coefficients of $(x_1 - x_{01})^{a_1}(x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}$, $c_{1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)}$, can be found from $\widetilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$, generally none of other terms, $c_{0_{a_0}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)}$, $a_0 \geq 1$ and $c_{0_{a_0}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(k)}$, $k \geq 0$, can be obtained from the differential equations in Theorem 2 because these coefficients depend on those indeterminate coefficients.

This indeterminacy problem does not arise when the volatility term is independent of t because those indeterminate terms turn out to be zero. For a reducible diffusion, if the dispersion matrix σ is a function of only x, the transformation γ would not be dependent on t either. Recall that the log-transition functions for irreducible diffusions have been postulated by using the results from reducible method. Let us go further and postulate the forms of $C_X^{(-1)}(t, x|t_0, x_0)$ and $C_X^{(0)}(t, x \mid x_0)$ as

$$C_X^{(-1)}(t, x|t_0, x_0) = C_X^{(-1)}(x|x_0) = -\frac{1}{2}\sum_{i=1}^{m}[\gamma_i(x) - \gamma_i(x_0)]^2 \tag{36}$$

and

$$C_X^{(0)}(t, x \mid x_0) = \sum_{i=1}^{m}[\gamma_i(x) - \gamma_i(x_0)] \int_0^1 \mu_{Y_i}\{t, \gamma(x_0) + u[\gamma(x) - \gamma(x_0)]\} du \tag{37}$$

by replacing y and y_0 with $\gamma(x)$ and $\gamma(x_0)$ as Proposition 2 states, respectively, in equations (17) and (18) even though there does not exist such γ for an irreducible diffusion process. Directly Taylor-expanding (36) and (37) about (t_0, x_0), $c^{(-1)}_{0_{a_0}1_{a_1}\cdots m_{a_m}} = 0$ and $c^{(0)}_{0_{a_0}} = 0$ for all $a_0 \geq 1$ and $|a| \geq 0$. Consequently, we can set $c^{(-1)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$ and $c^{(0)}_{0_{a_0}}$ to zeros for all $a_0 \geq 1$ and $|a| \geq 0$ to compute $\tilde{l}^{(K)}_X (t, x \mid t_0, x_0; \theta)$.

Instead, we can Taylor-expand $C^{(k)}_X (t, x \mid t_0, x_0)$ only in x up to j_k-order and write it as $\overline{C}^{(j_k,k)}_X (t, x \mid x_0)$,

$$\overline{C}^{(j_k,k)}_X (t, x \mid x_0) = \sum_{0 \leq |a| \leq j_k} c^{(k)}_{1_{a_1}\cdots m_{a_m}} (t, x_0) (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}, \tag{38}$$

where $c^{(k)}_{1_{a_1}\cdots m_{a_m}} (t, x_0)$ depends on t and x_0. Defining $\bar{l}^{(K)}_X (t, x \mid t_0, x_0)$ as

$$\bar{l}^{(K)}_X (t, x \mid t_0, x_0) = -\frac{m}{2} \ln (2\pi\Delta) - D_\upsilon (t, x) + \frac{\overline{C}^{(j_{-1},-1)}_X (t, x \mid x_0)}{\Delta} + \sum_{k=0}^{K} \overline{C}^{(j_k,k)}_X (t, x \mid x_0) \frac{\Delta^k}{k!}, \tag{39}$$

$\bar{l}^{(K)}_X$ can be found by using $\overline{f}^{(k-1)}_X (t, x \mid x_0)$ and $\overline{G}^{(k)}_X (t, x \mid x_0)$ which are obtained after $\overline{C}^{(j_k,k)}_X$ and Taylor-expansions of $\mu_i (t, x)$, $\upsilon_{ij} (t, x)$ and $D_\upsilon (t, x)$ in x only take the places of $C^{(k)}_X$, $\mu_i (t, x)$, $\upsilon_{ij} (t, x)$ and $D_\upsilon (t, x)$ in $f^{(k-1)}_X (t, x \mid t_0, x_0)$ and $G^{(k)}_X (t, x \mid t_0, x_0)$ in Theorem 2. In addition, $c^{(-1)}_{a_1\cdots a_m} (t, x_0)$ does not depend on t, so it is $c^{(-1)}_{1_{a_1}\cdots m_{a_m}} (x_0)$. Furthermore, $c^{(-1)}_{1_{a_1}\cdots m_{a_m}} (x_0) = 0$ when $|a| = 0$ and 1, and $c^{(0)}_{1_{a_1}\cdots m_{a_m}} (t, x_0)$ can be set to 0 for $|a| = 0$. Therefore, in order to find $\bar{l}^{(K)}_X$, we can follow exactly the same procedure as the time-homogeneous irreducible case except that now the coefficients of Taylor-expansion are functions of both x_0 and t. Thus, to make the approximation error equal to $O\left(\Delta^{K+1}\right)$, we can set $j_k = 2(K + 1 - k)$. In truth, the latter approach produces more accurate log-likelihood expansion than the former since $C^{(k)}_X (t, x \mid t_0, x_0)$ is not Taylor-expanded in t.

Nevertheless, if the elements of volatility matrix are functions of both t and x then $c^{(-1)}_{0_{a_0}h} \neq 0$ and $c^{(0)}_{0_{a_0}} \neq 0$ in general where $a_0 \geq 1$ and $h = 1, \cdots, m$ and we are unable to calculate any of the coefficients of Taylor-expansion $C^{(j_k,k)}_X (t, x \mid t_0, x_0)$, $k \geq -1$ but $c^{(-1)}_{1_{a_1}2_{a_2}\cdots m_{a_m}}$ of $C^{(j_{-1},-1)}_X (t, x|t_0, x_0)$. This is because the coefficients of Taylor-expansions $C^{(j_k,k)}_X (t, x \mid t_0, x_0)$ cannot be determined without knowing the coefficients of lower order terms and $f^{(k-1)}_X (t, x|t_0, x_0)$ pivots on the previous $C^{(j_h,h)}_X (x \mid x_0)$, $h \leq k-1$. In spite of this, if we look at each coefficient of $\Delta^{a_0} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}$ in $\tilde{l}^{(\infty)}_X (t, x \mid t_0, x_0)$ rather than that in $C^{(j_k,k)}_X (t, x \mid t_0, x_0)$, the indeterminacy problem is resolved as Theorem 3 says.

Theorem 3 *$\tilde{l}^{(\infty)}_X (t, x \mid t_0, x_0)$ is free from indeterminate terms, $c^{(-1)}_{0_{a_0}h}$ and $c^{(0)}_{0_{a_0}}$, where $a_0 \geq 1$ and $h = 1, \cdots, m$.*

The $c^{(0)}_{0_{a_0}} \Delta^{a_0}$ terms of $C^{(j_0,0)}_X (t, x \mid t_0, x_0)$ affect a_0 consecutive subsequent coefficients $C^{(j_k,k)}_X (t, x \mid t_0, x_0)$, $1 \leq k \leq a_0$ through only $\frac{\partial C^{(k-1)}_X (t,x|t_0,x_0)}{\partial t}$ in $G^{(k)}_X (t, x|t_0, x_0)$. Lemma A1 in Appendix proves that $c^{(0)}_{0_{a_0}}$ are cancelled out by these affected parts of $C^{(j_k,k)}_X (t, x \mid t_0, x_0)$ in $\tilde{l}^{(\infty)}_X$. Using this result and double mathematical induction on a_0 and $|a|$, where $a = (a_1, \cdots, a_m)$, the indeterminate parts of the coefficients of $C^{(j_k,k)}_X (t, x \mid t_0, x_0)$ can be shown to be offset in $\tilde{l}^{(\infty)}_X (t, x \mid t_0, x_0)$ despite the fact that all of $C^{(j_k,k)}_X (t, x \mid t_0, x_0)$'s of $\tilde{l}^{(\infty)}_X$ are incomputable on account of the indeterminate terms $c^{(-1)}_{0_{a_0}h}$. Precisely speaking, the coefficient of $\Delta^{a_0} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}$ in $\tilde{l}^{(\infty)}_X (t, x \mid t_0, x_0)$ is $\sum_{k=-1}^{a_0} c^{(k)}_{0_{a_0-k}1_{a_1}\cdots m_{a_m}} \frac{1}{k!}$, which is free of the indeterminate terms although each of $c^{(k)}_{0_{a_0-k}1_{a_1}\cdots m_{a_m}}$ is not. For this reason, the indeterminacy of the coefficients, $c^{(-1)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$, $a_0 \neq 0$, of $\Delta^{a_0}(x_1-x_{01})^{a_1}(x_2-x_{02})^{a_2}\cdots(x_m-x_{0m})^{a_m}$ in $C^{(j_{-1},-1)}_X(t,x \mid t_0,x_0)$ and that of all subsequent coefficients due to their dependence on $c^{(-1)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$ is not a problem in computing $\widetilde{l}^{(K)}_X(t,x \mid t_0,x_0)$ as long as we include first a_0+2 coefficients of $\widetilde{l}^{(\infty)}_X$, that is, $C^{(j_k,k)}_X(t,x \mid t_0,x_0)$, $-1 \leq k \leq a_0$ in $\widetilde{l}^{(\infty)}_X$. The pairs of orders $(a_0,|a|)$ for each $C^{(j_k,k)}_X$ to have approximation error equal to $O_p\left(\Delta^{K+1}\right)$ are tabulated in (33). In reality, we can compute not $\widetilde{l}^{(\infty)}_X$ but $\widetilde{l}^{(K)}_X$. Even so, $\widetilde{l}^{(K)}_X$ does not suffer from the indeterminacy problem once we find $C^{(j_k,k)}_X$ according to $(a_0,|a|)$ in (33) because the indeterminable parts of included $C^{(j_k,k)}_X$ are eliminated in $\widetilde{l}^{(K)}_X$. We need to derive $\widetilde{l}^{(K)}_X$, and yet the constant term of $C^{(j_{K+1},K+1)}_X$ must be added so that the indeterminate terms can be removed from $\widetilde{l}^{(K)}_X$. Note that when $|a|=0$, a_0 of $c^{(-1)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$ must be $K+2$ as can be seen from (33) so we need to include first $K+3$ coefficients of $\widetilde{l}^{(\infty)}_X$.

Let us see how to obtain $\widetilde{l}^{(K)}_X(t,x \mid t_0,x_0)$ in practice. The constant and first order terms of $C^{(j_{-1},-1)}_X(t,x \mid t_0,x_0)$ are zero. Remember that Theorem 3 says that we can set $c^{(-1)}_{0_{a_0}h}=0$ and $c^{(0)}_{0_{a_0}}=0$ for all $a_0 \geq 1$ and $h=1,\cdots,m$ since these are cancelled out in $\widetilde{l}^{(K)}_X$ although they are not zeros. So, to find $C^{(j_{-1},-1)}_X(t,x \mid t_0,x_0)$, first we can put zeros in $c^{(-1)}_{0h}$ and $c^{(-1)}_{0_2}$, where $h=1,\cdots,m$ for the reason that $c^{(-1)}_{0_{a_0+1}}$ depends on only $c^{(-1)}_{0_{a_0}h}$. Then, we get the coefficients of the second order terms as (34).

Once $c^{(-1)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$ $0 \leq a_0+a_1+\cdots a_m \leq l$ for a combination $(a_0,|a|)$ in (33) are acquired and put in $C^{(j_{-1},-1)}_X(t,x \mid t_0,x_0)$, $c^{(-1)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$ for the next pair $(a_0,|a|)$ in (33) with $a_0+a_1+\cdots a_m=l+1$, can be found by solving a system of linear equations obtained from equating all corresponding coefficients of the $l+1$-th order terms of $\widetilde{f}^{(-2)}_X(t,x \mid t_0,x_0)$ to zero. In doing so, make sure to set $c^{(-1)}_{0_l h}=0$. Next, since the constant term of $C^{(j_0,0)}_X(t,x \mid t_0,x_0)$ is zero, given $C^{(j_{-1},-1)}_X(t,x \mid t_0,x_0)$, we set the first order terms of $\widetilde{f}^{(-1)}_X(t,x \mid t_0,x_0)$ to zero to find $c^{(0)}_h, h=1,\cdots,m$, by solving the resultant system of linear equations. Remember that $c^{(0)}_0=0$. Similarly, with $C^{(j_{-1},-1)}_X(t,x \mid t_0,x_0)$ and $c^{(0)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$, $0 \leq a_0+a_1+\cdots a_m \leq l$ for a couple $(a_0,|a|)$ in (33), $c^{(0)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$, $a_0+a_1+\cdots a_m=l+1$ for the next combination $(a_0,|a|)$ in (33) can be calculated. The same principle applies to all subsequent coefficients. That is to say, once we find $C^{(j_{-1},-1)}_X(t,x \mid t_0,x_0),\cdots,C^{(j_{k-1},k-1)}_X(t,x \mid t_0,x_0)$, and $c^{(k)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$, $0 \leq a_0+a_1+\cdots a_m \leq l$ with $(a_0,|a|)$ in (33), we can compute $c^{(k)}_{0_{a_0}1_{a_1}\cdots m_{a_m}}$, $a_0+a_1+\cdots a_m=l+1$ for the next couple $(a_0,|a|)$ in (33) successively up to desired order.

Because the indeterminate coefficients, $c^{(-1)}_{0_{a_0+1}}$ and $c^{(-1)}_{0_{a_0}h}$, where $a_0 \geq 1$ and $h=1,\cdots,m$, correspond to the coefficients of Taylor expansions of $c^{(-1)}_{a_1\cdots a_m}(t,x_0)$ in $(t-t_0)$ for $|a|=0$ and 1, respectively, and $c^{(0)}_{0_{a_0}}$ to those of $c^{(0)}_{a_1\cdots a_m}(t,x_0)$ in $(t-t_0)$ for $|a|=0$ where $a_0 \geq 1$, and they are cancelled out in $\widetilde{l}^{(\infty)}_X$, we can let $c^{(-1)}_{a_1\cdots a_m}(t,x_0)=0$ for $|a|=0$ and 1 and $c^{(0)}_{a_1\cdots a_m}(t,x_0)=0$ for $|a|=0$ when computing $\bar{l}^{(\infty)}_X$. However, we can find only $\bar{l}^{(K)}_X$ in practice so we cannot follow this alternative approach.

4 Log-likelihood Function and its Convergence

Even though model (2) is continuous time model, we can observe X_t only at discrete times such that $\{X_t | t = i\Delta \text{ and } i = 0, \cdots, n\}$, where Δ is the fixed time interval between two consecutive observations. Due to the first order Markov property of model (2), the log-likelihood function of data can be written as

$$l_n(\theta) = \sum_{i=1}^{n} l_X\left(i\Delta, X_{i\Delta} | (i-1)\Delta, X_{(i-1)\Delta}; \theta\right),$$

where the log-density of the initial observation X_0 is ignored because it is asymptotically negligible. Replacing l_X with the series expansion of log-transition density function $l_X^{(K)}$ ($\widetilde{l}_X^{(K)}$) gives approximate log-likelihood function $l_n^{(K)}(\theta)$ ($\widetilde{l}_n^{(K)}(\theta)$). It can be shown that $l_X^{(K)}$ and $\widetilde{l}_X^{(K)}$ converge to l_X and therefore $l_n^{(K)}(\theta)$ and $\widetilde{l}_n^{(K)}(\theta)$ converge to $l_n(\theta)$.

Assume that $\mu_i(t, x; \theta)$ and $\sigma_{ij}(t, x; \theta)$, $i, j = 1, \cdots, m$, and their derivatives at all orders are thrice differentiable in θ. Let θ_0 be the true values of the parameter vector θ. For fixed n and Δ, suppose that $l_n(\theta)$ has a unique MLE $\hat{\theta}_{n,\Delta} \in \Theta$ for θ. The notation $\hat{\theta}_{n,\Delta}^{(K)}$ denotes the approximate MLE of θ computed by maximizing $l_n^{(K)}(\theta)$ ($\widetilde{l}_n^{(K)}(\theta)$).

Theorem 4 *For all n,*

$$\sup_{\theta \in \Theta} \left| \widetilde{l}_n^{(K)}(\theta) - l_n(\theta) \right| \to 0 \tag{40}$$

in P_{θ_0}-probability as $\Delta \to 0$. In the reducible case, the same holds for $l_n^{(K)}(\theta)$. The approximate MLE $\hat{\theta}_{n,\Delta}^{(K)}$ exists almost surely and satisfies $\hat{\theta}_{n,\Delta}^{(K)} - \hat{\theta}_{n,\Delta} \to 0$ in P_{θ_0}-probability as $\Delta \to 0$.

Furthermore, suppose that as $n \to \infty$, we have $\hat{\theta}_{n,\Delta}^{(K)} - \theta_0 \to 0$ in P_{θ_0}-probability and that there exists a sequence of nonsingular $p \times p$ matrices $S_{n,\Delta}$ such that

$$S_{n,\Delta}^{-1}\left(\hat{\theta}_{n,\Delta} - \theta_0\right) = O_p(1).$$

Then, there exists a sequence $\Delta_n \to 0$ such that

$$S_{n,\Delta_n}^{-1}\left(\hat{\theta}_{n,\Delta_n}^{(K)} - \hat{\theta}_{n,\Delta_n}\right) = o_p(1). \tag{41}$$

Because $l_X^{(K)}$ ($\widetilde{l}_X^{(K)}$) is a Taylor-expansion of l_X about $\Delta = 0$ (and about $x = x_0$), the approximation error in (40) is inconsiderable in a small neighborhood of x_0. Even if l_X is not analytic outside of the neighborhood, it doesn't cause a big problem because the approximation error is at most polynomial while the probability of X_t being in this region in time Δ is exponentially small.

5 Accuracy of Reducible and Irreducible Methods

5.1 Extended Cox-Ingersoll-Ross (ECIR) model

Consider the Extended Cox-Ingersoll-Ross (ECIR) model

$$dX_t = a\left(b_t - X_t\right)dt + \sigma_0 e^{\sigma_1 t}\sqrt{X_t}dW_t \tag{42}$$

where $b_t = \left(\sigma_0^2 d/4a\right)\exp\left(2\sigma_1 t\right)$ and d is a positive integer. True transition density function of this ECIR model is known (Maghsoodi (1996)) to be

$$p_X\left(t, x|t_0, x_0\right) = \frac{1}{2}G\left(t, t_0\right)\exp\left(-\frac{1}{2}\left(\lambda\left(t, t_0\right) + G\left(t, t_0\right)x\right)\right)\left(\frac{G\left(t, t_0\right)x}{\lambda\left(t, t_0\right)}\right)^{(d-2)/4} I_{(d-2)/2}\left(\sqrt{\lambda\left(t, t_0\right)G\left(t, t_0\right)x}\right) \tag{43}$$

where $G\left(t, t_0\right) = \left(4a/\sigma_0^2\right)\exp\left(-2\sigma_1 t\right)/\left[1 - \exp\left(-a\left(t - t_0\right)\right)\right]$, $I_\alpha\left(\cdot\right)$ is the modified Bessel function of the first kind of order α, and $\lambda\left(t, t_0\right) = 4x_0\left(2\sigma_1 + a\right)/\left[\left(\sigma_0^2\right)\left(\exp\left(2\sigma_1 t + \left(t - t_0\right)a\right) - \exp\left(2\sigma_1 t_0\right)\right)\right]$.

Table 1 summarizes Monte Carlo simulation study results for (42). True transition density function of the ECIR process (43) has been used to generate 1000 samples of 500 weekly observations with the same initial condition $x_0 = 0.05625$ and parameter values $a = 5$, $\sigma_0 = 0.15$ and $\sigma_1 = 0.001$. Using the true density function and two approximate density functions obtained respectively by reducible and irreducible methods for the case $K = 2$, we estimate parameters by MLE. Denote parameter estimates when using the true density as $\hat{\theta}^{(True)}$, the reducible method as $\hat{\theta}^{(Re)}$, and the irreducible method as $\hat{\theta}^{(Irre)}$. Sample means and standard deviations of $\hat{\theta}^{(Re)} - \hat{\theta}^{(True)}$ and $\hat{\theta}^{(Irre)} - \hat{\theta}^{(True)}$ are much smaller than those of $\hat{\theta}^{(True)} - \theta_0$, where θ_0 is the true parameter. Our simulation studies suggest that both the reducible and irreducible methods provide a good approximation to the true density function of model (42) and it can be used when the true density is unknown as is often the case.

Table 1: Simulation Study of ECIR Model

(1000 samples of 500 weekly data with $x_0 = 0.05625$)							
Parameter	θ_0	$\hat{\theta}_{ML}^{(True)} - \theta_0$		$\hat{\theta}_{ML}^{(True)} - \hat{\theta}_{ML}^{(2,Re)}$		$\hat{\theta}_{ML}^{(True)} - \hat{\theta}_{ML}^{(2,Irre)}$	
		Mean	Std. Dev	Mean	Std. Dev	Mean	Std. Dev
a	0.5	6.7×10^{-2}	5.9×10^{-2}	-2.7×10^{-3}	1.0×10^{-4}	-6.1×10^{-3}	3.1×10^{-3}
σ_0	0.15	-1.7×10^{-4}	9.0×10^{-5}	7.6×10^{-6}	7.0×10^{-9}	-3.6×10^{-4}	3.3×10^{-5}
σ_1	0.001	5.1×10^{-4}	1.3×10^{-4}	-2.5×10^{-5}	4.2×10^{-8}	-1.4×10^{-5}	7.6×10^{-6}

5.2 Bivariate Time-Inhomogeneous Ornstein-Uhlenbeck Model

Now let's have a look at multivariate time-inhomogeneous models to investigate the performance of the reducible and irreducible methods. Consider the d-dimensional diffusion process following

$$dX_t = \left[A\left(t\right)X_t + a\left(t\right)\right] + B\left(t\right)dW_t, \tag{44}$$

where $A(t)$ and $B(t)$ are $d \times d$ matrices and $a(t)$ is a $d \times 1$ vector with each element possibly being a function of time. The transition probability of X_t is known to have a d-dimensional normal distribution (see page 149 in Arnold (1974)),

$$N\left[m_t(t_0, x_0), V_t(t_0, x_0)\right],$$

where the mean is

$$m_t(t_0, x_0) = C(t, t_0)\left[x_0 + \int_{t_0}^{t} C(u, t_0)^{-1} a(u)\, du\right]$$

and the variance is

$$V_t(t_0, x_0) = C(t, t_0)\left[\int_{t_0}^{t} C(u, t_0)^{-1} B(u) B(u)^T \left[C(u, t_0)^{-1}\right]^T du\right] C(t, t_0)^T.$$

$C(t, t_0)$ is the solution of the homogeneous matrix equation

$$\frac{d}{dt} C(t, t_0) = A(t) C(t, t_0),\ C(t_0, t_0) = I_d.$$

Let $\alpha = [\alpha_i]_{i=1,2}$, $\beta = [\beta_i]_{i=1,2}$, and $k = [k_{i,j}]_{i,j=1,2}$ which is assumed to be full rank. It can be shown that the bivariate time-inhomogeneous Ornstein-Uhlenbeck process,

$$dY_t = k(\alpha + \beta t - Y_t)\, dt + dW_t \tag{45}$$

has the bivariate normal distribution with mean

$$m_t(t_0, y_0) = \exp(-k\Delta)\left(y_0 - \alpha - \beta t_0 + \int_0^{\Delta} \exp(ku)\, k\beta u du\right) + \alpha + \beta t_0$$

and variance-covariance

$$V_t(t_0, y_0) = \int_0^{\Delta} \exp\left[-k(\Delta - u)\right] \left(\exp\left[-k(\Delta - u)\right]\right)^T du.$$

The parameters in (45) cannot be identified on the basis of discrete data. Phillips (1973), Hansen and Sargent (1983) and Kessler and Rahbek (2004) discuss this problem. We will assume that $k_{21} = 0$ then k becomes a triangular matrix with real elements and it can have real eigenvalues. Then the mapping $k \to \exp(-k\Delta)$ is invertible and the parameters can be identified from discrete data. The process Y_t is a unit diffusion so we can use reducible method to find the coefficients of the likelihood expansion.

To my best knowledge (44) is the only multivariate model for which true transition density function is available. We transform Y_t into X_t using $(X_{1t}, X_{2t})^T = (\exp(Y_{1t}), \exp(Y_{2t}))^T$ to apply the irreducible method directly to the transformed process. Applying Ito's lemma, the resulting process is

$$dX_t = \begin{pmatrix} X_{1t}\left[\frac{1}{2} + k_{11}(\alpha_1 + \beta_1 t - \ln(X_{1t})) + k_{12}(\alpha_2 + \beta_2 t - \ln(X_{2t}))\right] \\ X_{2t}\left[\frac{1}{2} + k_{21}(\alpha_1 + \beta_1 t - \ln(X_{1t})) + k_{22}(\alpha_2 + \beta_2 t - \ln(X_{2t}))\right] \end{pmatrix} dt + \begin{pmatrix} X_{1t} & 0 \\ 0 & X_{2t} \end{pmatrix} dW_t. \tag{46}$$

True log-transition function of X_t can be obtained by change of variable because the transition density of Y_t is known. Therefore

$$l_X(t, x | t_0, x_0) = -\ln(x_1 x_2) + l_Y\left[t, \ln(x_1), \ln(x_2) | t_0, \ln(x_{01}), \ln(x_{02})\right]$$

can be used as a benchmark.

From the true transition density of X_t, 1000 samples of 500 weekly observations ($\Delta = 1/52$) were sampled. For each sample, we have found maximum likelihood estimates of the parameters. Similarly to Table 1, sample means and standard deviations of $\hat{\theta}^{(Re)} - \hat{\theta}^{(True)}$, $\hat{\theta}^{(Irre)} - \hat{\theta}^{(True)}$ and $\hat{\theta}^{(True)} - \theta_0$ are computed as tabulated in Table 2. Again both the reducible and irreducible methods yield approximate log-likelihood functions that can be used for the true log-density function when it is not available.

Table 2: Simulation Study

(1000 samples of 500 weekly data with $x_{01} = 0.0002$ and $x_{02} = 0.0004$)							
Parameter	θ_0	$\hat{\theta}_{ML}^{(True)} - \theta_0$		$\hat{\theta}_{ML}^{(True)} - \hat{\theta}_{ML}^{(2,\ Redu)}$		$\hat{\theta}_{ML}^{(True)} - \hat{\theta}_{ML}^{(2,\ Irre)}$	
		Mean	Std. Dev	Mean	Std. Dev	Mean	Std. Dev
α_1	0	-7.6×10^{-3}	1.6×10^{-2}	4.2×10^{-7}	3.6×10^{-10}	6.6×10^{-5}	2.4×10^{-8}
α_2	0	8.7×10^{-4}	4.2×10^{-3}	-3.6×10^{-6}	3.0×10^{-10}	3.7×10^{-5}	2.4×10^{-9}
β_1	0.001	2.0×10^{-3}	5.2×10^{-4}	-4.1×10^{-7}	2.7×10^{-11}	1.2×10^{-8}	9.3×10^{-10}
β_2	0.005	-1.5×10^{-4}	1.3×10^{-4}	2.1×10^{-7}	2.6×10^{-11}	6.8×10^{-8}	1.5×10^{-10}
k_{11}	5	0.75	1.37	1.4×10^{-2}	7.9×10^{-5}	1.3×10^{-2}	2.2×10^{-4}
k_{12}	1	0.14	2.87	1.1×10^{-2}	3.3×10^{-4}	1.5×10^{-3}	2.1×10^{-5}
k_{22}	10	0.68	2.44	7.5×10^{-2}	1.1×10^{-3}	1.9×10^{-2}	5.2×10^{-4}

6 Conclusion

Diffusion models have been extensively used in modeling the dynamics of many economic variables. Researchers propose multivariate time-inhomogeneous diffusion models to capture time dependency of the underlying data generating process. This paper suggests a way to find a very accurate approximate transition density function of a multivariate time-inhomogeneous diffusion in closed-form. Then, using our method, we can get the maximum likelihood estimator which is asymptotically efficient in the class of all consistent and uniformly asymptotically normal estimators.

References

Ahn, D.-H., R. F. Dittmar, and A. Gallant (2002): "Quadratic Term-Structure Models: Theory and Evidence," *Review of Financial Studies*, 15(1), 243–288.

Aït-Sahalia, Y. (1999): "Transition Densities for Interest Rate and Other Nonlinear Diffusions," *Journal of Finance*, 54, 1361–1395.

——— (2002): "Maximum Likelihood Estimation of Discretely Sampled Diffusions: A Closed-Form Approximation Approach," *Econometrica*, 70(1), 223–262.

——— (2008): "Closed-Form Likelihood Expansions for Multivariate Diffusions," *Annals of Statistics*, 36(2), 906–937.

Andersen, T., and J. Lund (1997): "Estimating Continuous-Time Stochastic Volatility Models of the Short-Term Interest Rate," *Journal of Econometrics*, 77(2), 343–377.

Andersen, T. G., and T. Bollerslev (1997): "Intraday periodicity and volatility persistence in financial markets," *Journal of Empirical Finance*, 4, 115–158.

——— (1998): "Deutsche Mark-Dollar Volatility: Intraday Activity Patterns, Macroeconomic Announcements, and Longer Run Dependencies," *Journal of Finance*, 53(1), 219–265.

Andersen, T. G., T. Bollerslev, F. X. Diebold, and C. Vega (2003): "Micro Effects of Macro Announcements: Real-Time Price Discovery in Foreign Exchange," 93(1), 38–62.

Arapis, M., and J. Gao (2006): "Empirical Comparisons in Short-term Interest Rate Models Using Nonparametric Methods," *Journal of Financial Econometrics*, 4, 310–345.

Arnold, L. (1974): *Stochastic Differential Equations: Theory and Applications*. John Wiley & Sons, New York.

Bartolini, L., G. Bertola, and A. Prati (2002): "Day-to-Day Monetary Policy and the Volatility of the Federal Funds Interest Rate," *Journal of Money, Credit and Banking*, 34(1), 137–159.

Bibby, B. M., and M. Sørenson (1995): "Martingale Estimation Functions for Discretely Observed Diffusion Processes," *Bernoulli*, 1, 17–39.

Black, F., and P. Karasinski (1991): "Bond and Option Pricing When Short Rates are Lognormal," *Financial Analysts Journal*, pp. 52–59.

Bollerslev, T., J. Cai, and F. M. Song (2000): "Intraday periodicity, long memory volatility, and macroeconomic announcement effects in the US Treasury bond market," *Journal of Empirical Finance*, 7, 37–55.

Bolton, P., and C. Harris (1999): "Strategic Experimentation," *Econometrica*, 67(2), 349–374.

Brennan, M., and E. Schwartz (1979): "A Continuous-Time Approach to the Pricing of Bonds," *Journal of Banking and Finance*, 3(2), 133–155.

Chen, R.-R., and L. Scott (1993): "Maximum Likelihood Estimation for a Multifactor Equilibrium Model of the Term Structure of Interest Rates," *Journal of Fixed Income*, 3(3), 14–31.

Choi, S. (2009): "Regime-Switching Univariate Diffusion Models of the Short-Term Interest Rate," *Studies in Nonlinear Dynamics & Econometrics*, 13(1), Article 4.

Chung, K. L., and R. J. Williams (1990): *Introduction to Stochastic Integration*. Birkhäuser Boston.

Dai, Q., and K. J. Singleton (2000): "Specification Analysis of Affne Term-Structure Models," *Journal of Finance*, 55(5), 1943–78.

Derman, E., and I. Kani (1994): "The Volatility Smile and Its Implied Tree," *Quantitative Strategies Research Notes*, Goldman Sachs.

Dixit, A. K., and R. S. Pindyck (1994): *Investment under Uncertainty*. Princeton University Press.

Duffie, D. (2001): *Dynamic asset pricing theory*. Princeton University Press, 3rd ed.

Duffie, D., and P. Glynn (2004): "Estimation of Continuous-Time Markov Processes Sampled at Random Time Intervals," *Econometrica*, 72(6), 1773–1808.

Duffie, D., and R. Kan (1996): "A Yield-Factor Model of Interest Rates," *Mathematical Finance*, 6(4), 379–406.

Duffie, D., and K. Singleton (1993): "Simulated Moments Estimation of Markov Models of Asset Prices," *Econometrica*, 61, 929–952.

Durham, G. B., and A. R. Gallant (2002): "Numerical Techniques for Simulated Maximum Likelihood Estimation of Continuous-Time Diffusion," *Journal of Business and Economic Statistics*, 20, 297–338.

Egorov, A. V., H. Li, and Y. Xu (2003): "Maximum Likelihood Estimation of Time Inhomogeneous Diffusions," *Journal of Econometrics*, 114, 107–139.

Elerian, O., S. Chib, and N. Shephard (2001): "Likelihood Inference for Discretely Observed Non-linear Diffusions," *Econometrica*, 69(4), 959–993.

Eraker, B. (2001): "MCMC Analysis of Diffusion Models with Application to Finance," *Journal of Business and Economic Statistics*, 19, 177–191.

Filipović, D. (2005): "Time-Inhomogeneous Affine Processes," *Stochastic Processes and Their Applications*, 115(4), 639–659.

Franses, P. H. (1996): "Recent Advances in Modelling Seasonality," *Journal of Economic Surveys*, 10(3), 299–345.

Friedman, A. (1975): *Stochastic Differential Equations and Applications*, vol. 1. Academic Press.

Froot, K. A., and M. Obstfeld (1991): "Exchange-Rate Dynamics under Stochastic Regime Shifts, A Unified Approach," *Journal of International Economics*, 31, 203–229.

Gallant, A. R., and G. Tauchen (1998): "Reprojecting Partially Observed Systems with Application to Interest Rate Diffusions," *Journal of the American Statistical Association*, 93(441), 10–24.

Gouriéroux, C. S., A. Monfort, and E. Renault (1993): "Indirect Inference," *Journal of Applied Econometrics*, 8, S85–S118.

Hansen, L. P., and T. J. Sargent (1983): "The Dimensionality of the Aliasing Problem in Models with Rational Spectral Densities," *Econometrica*, 51, 377–387.

Hansen, L. P., and J. A. Scheinkman (1995): "Back to the Future: Generating Moment Implications for Continuous-Time Markov Processes," *Econometrica*, 63(4), 767–804.

Hansen, P. R., and A. Lunde (2005): "Testing the Significance of Calendar Effects," *Federal Reserve Bank of Atlanta*, Working Paper 2005-2.

Heston, S. L. (1993): "A Closed-Form Solution for Options with Stochastic Volatility with Applications to Bond and Currency Options," *Review of Financial Studies*, 6(2), 327–343.

Ho, T. s., and S.-B. Lee (1986): "Term Structure Movements and Pricing Interest Rate Contingent Claims," *Journal of Finance*, 41(5), 1011–1029.

Holmstrom, B., and P. Milgrom (1987): "Aggregation and Linearity in the Provision of Intertemporal Incentives," *Econometrica*, 55(2), 303–328.

Hull, J., and A. White (1990): "Pricing Interest-Rate-Derivative Securities," *Review of Financial Studies*, 3(4), 573–592.

——— (1994): "Numerical Procedures for Implementing Term Structure Models II: Two-Factor Models," *Journal of Derivatives*, 2(2), 37–48.

J., L. J., and E. S. Schwartz (2002): "Electricity Prices and Power Derivatives: Evidence from the Nordic Power Exchange," *Review of Derivatives Research*, 5, 5–50.

Jones, C. S. (2003): "Nonlinear Mean Reversion in the Short-Term Interest Rate," *Review of Financial Studies*, 16(3), 793–843.

Jordan, S. D., and B. D. Jordan (1991): "Seasonality in Daily Bond Returns," *Journal of Financial and Quantitative Analysis*, 26(2), 269–285.

Karatzas, I., and S. E. Shreve (1998): *Brownian Motion and Stochastic Calculus.* Springer-Verlag, New York, NY, 2nd edition.

Kessler, M., and A. Rahbek (2004): "Identification and Inference for Multivariate Cointegrated and Ergodic Gaussian Diffusions," *Statistical Inference Stochastic Processes*, 7, 137–151.

Kessler, M., and M. Sørenson (1999): "Estimating Equations Based on Eingenfunctions for a Discretely Observed Diffusion," *Bernoulli*, 5, 299–314.

Langetieg, T. C. (1980): "A Multivariate Model of the Term Structure," *Journal of Finance*, 35(1), 71–97.

Lo, A. W. (1988): "Maximum Likelihood Estimation of Generalized Ito Processes with Discretely Sampled Data," *Econometric Theory*, 4, 231–247.

Lockwood, L. J., and S. C. Linn (1990): "An Examination of Stock Market Return Volatility During Overnight and Intraday Periods, 1964-1989," *Journal of Finance*, 45(2), 591–601.

McCullagh, P. (1987): *Tensor Methods in Statistics.* Chapman and Hall, London, U.K.

Melino, A. (1994): *Estimation of Continuous-Time Models in Finance*, vol. II. Cambridge University Press.

Merton, R. (1980): "On Estimating the Expected Return on the Market: An Exploratory Investigation," *Journal of Financial Economics*, 8, 323–361.

Misiorek, A., S. Trueck, and R. Weron (2006): "Point and Interval Forecasting of Spot Electricity Prices: Linear vs. Non-Linear Time Series Models," *Studies in Nonlinear Dynamics & Econometrics*, 10(3), Article 2.

Pearson, N. D., and T.-S. Sun (1994): "Exploiting the Conditional Density in Estimating the Term Structure: An Application to the Cox, Ingersoll, and Ross Model," *Journal of Finance*, 49(4), 1279–1304.

Pedersen, A. R. (1995): "A New Approach to Maximum-Likelihood Estimation for Stochastic Differential Equations Based on Discrete Observations," *Scandinavian Journal of Statistics*, 22, 55–71.

Phillips, P. C. (1973): "The Problem of Identification in Finite Parameter Continuous Time Models," *Journal of Econometrics*, 1, 351–362.

——— (2001): "Trending Time Series and Macroeconomic Activity: Some Present and Future Challanges," *Journal of Econometrics*, 100, 21–27.

Piazzesi, M. (2005): "Bond Yields and the Federal Reserve," *Journal of Political Economy*, 113(2), 311–344.

Purzitsky, A. (2003): "Closed Form Maximum Likelihood Estimation of Discretely Sampled Jump-Diffusions," *Ph.D. thesis*, Princeton University.

Rubinstein, M. (1994): "Implied Binomial Trees," *Journal of Finance*, 49(3), 771–818.

Santa-Clara, P. (1995): "Simulated Likelihood Estimation of Diffusions with an Application to the Short Term Interest Rate," *Ph.D. thesis*, INSEAD.

Schaumburg, E. (2001): "Maximum Likelihood Estimation of Jump Processes," *Ph.D. thesis*, Princeton University.

Skorokhod, A. V. (1965): *Studies in the Theory of Random Processes*. Addison-Wesley Publishing Company, Inc.

Sonoc, C. (1998): "On the Pathwise Uniqueness of Solutions of Stochastic Differential Equations," *Portugaliae Mathematica*, 55, 451–456.

Stambaugh, R. (1988): "The Information in Forward Rates," *Journal of Financial Economics*, 21(1), 41–70.

Stanton, R. (1997): "A Nonparametric Model of Term Structure Dynamics and the Market Price of Interest Rate Risk," *Journal of Finance*, 52(5), 1973–2002.

Stroock, D. W., and S. S. Varadhan (1969): "Diffusion Processes with Continuous Coefficients, I," *Communications on Pure and Applied Mathematics*, XXII, 345–400.

——— (1979): *Multidimensional Diffusion Processes*. Springer, New York.

Swart, J. M. (2001): "A 2-Dimensional SDE Whose Solutions Are Not Unique," *Electronic Communications in Probability*, 6, 67–71.

Withers, C. S. (2000): "A Simple Expression for the Multivariate Hermite Polynomials," *Statistics and Probability Letters*, 47, 165–169.

Yamada, T., and S. Watanabe (1971a): "On the Uniqueness of Solutions of Sotchastic Differential Equations," *J. Math. Kyoto Univ.*, 11(1), 155–167.

——— (1971b): "On the Uniqueness of Solutions of Sotchastic Differential Equations," *J. Math. Kyoto Univ.*, 11(3), 553–563.

Yu, J. (2007): "Closed-Form Likelihood Approximation and Estimation of Jump-Diffusions with an Applicaiton to the Realignment Risk of the Chinese Yuan," *Journal of Econometrics*, 141, 1245–1280.

Zvonkin, A. K., and N. Krylov (1981): "On Strong Solutions of Stochastic Differential Equations," *Selecta Mathematica Sovietica*, 1(1), 19–61.

A Proofs

Proof of Proposition 1

Proof. Because the reducibility condition has to do with only the state variables vector x not t, Proposition 1 can be proved in the same way as Proposition 1 in Aït-Sahalia (2008). ■

Proof of Theorem 1

Proof. Let $F_Y^{(K)}(t, y|t_0, y_0)$ and $B_Y^{(K)}(t, y|t_0, y_0)$ be, respectively, the difference between the left- and right-hand sides of (22) and (23) when we substitute (16) for $l_Y(t, y \mid t_0, y_0)$. It is well known that the log-transition density function of a diffusion process is the solution of the Kolmogorov forward and backward partial differential equations. Therefore, it is sufficient to verify that $F_Y^{(K)}(t, y|t_0, y_0)$ and $B_Y^{(K)}(t, y|t_0, y_0)$ are of order Δ^K in order to prove Theorem 1. Collecting terms of $F_Y^{(K)}(t, y|t_0, y_0)$ and $B_Y^{(K)}(t, y|t_0, y_0)$ in increasing orders of Δ, we obtain

$$F_Y^{(K)}(t, y|t_0, y_0) = \sum_{k=-2}^{K-1} f_Y^{(k)}(t, y|t_0, y_0) \frac{\Delta^k}{k!} + O\left(\Delta^K\right) \tag{47}$$

and

$$B_Y^{(K)}(t, y|t_0, y_0) = \sum_{k=-2}^{K-1} b_Y^{(k)}(t, y|t_0, y_0) \frac{\Delta^k}{k!} + O\left(\Delta^K\right) \tag{48}$$

with the convention that $(-2)! = (-1)! = 0! = 1$. We will show that $f_Y^{(k)}(t, y|t_0, y_0) = 0$ and $b_Y^{(k)}(t, y|t_0, y_0) = 0$ for all $k = -2, -1, \cdots, K-1$ are satisfied by the coefficients given in Theorem 1. In fact, those coefficients can be found by solving the partial differential equations of the coefficients, $f_Y^{(k)}(t, y|t_0, y_0) = 0$ and $b_Y^{(k)}(t, y|t_0, y_0) = 0$.

First,

$$C_Y^{(-1)}(t, y \mid t_0, y_0) = -\frac{1}{2} \sum_{i=1}^{m} \left[\frac{\partial C_Y^{(-1)}(t, y \mid t_0, y_0)}{\partial y_i} \right]^2 \tag{49}$$

from $f_Y^{(-2)}(t, y|t_0, y_0) = 0$. Since the density goes to a Gaussian as $\Delta \to 0$, the solution $C_Y^{(-1)}(t, y \mid t_0, y_0)$ to (49) must have a strict maximum at $y = y_0$ such that

$$C_Y^{(-1)}(t, y \mid t_0, y_0) = -\frac{1}{2} \sum_{i=1}^{m} (y_i - y_{0i})^2 . \tag{50}$$

Given (50),

$$f_Y^{(-1)}(t, y|t_0, y_0) = -\sum_{i=1}^{m} (y_i - y_{0i}) \mu_{Y_i}(t, y) + \sum_{i=1}^{m} (y_i - y_{0i}) \frac{\partial C_Y^{(0)}(t, y \mid t_0, y_0)}{\partial y_i}.$$

Solving $f_Y^{(-1)}(t, y|t_0, y_0) = 0$ for $C_Y^{(0)}(t, y \mid y_0, s)$,

$$C_Y^{(0)}(t, y \mid t_0, y_0) = \sum_{i=1}^{m} (y_i - y_{0i}) \int_0^1 \mu_{Y_i}[t, y_0 + u(y - y_0)] \, du + \sum_{i,j=1, j \neq i}^{m} \alpha_{ij}^{(0)} \frac{y_i - y_{0i}}{y_j - y_{0j}} + M_Y^{(0)},$$

where $\alpha_{ij}^{(0)}$ and $M_Y^{(0)}$ are independent of y because they are integration constants. Moreover $\alpha_{ij}^{(0)} = 0$ for all $i, j = 1, \cdots, m$ $(j \neq i)$ due to the boundary condition that $C_Y^{(0)}(t, y \mid t_0, y_0)$ is finite when $y_j = y_{0j}$ for all $j = 1, \cdots, m$. Since $C_Y^{(0)}(t, y \mid t_0, y_0)$ should also satisfy

$$b_Y^{(-1)}(t, y|t_0, y_0) = -\sum_{i=1}^m \mu_{Y_i}(t, y)(y_i - y_{0i}) - \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial C_Y^{(0)}(t, y \mid t_0, y_0)}{\partial y_{0i}} = 0$$

it must be

$$\sum_{i=1}^m (y_i - y_{0i}) \frac{\partial M_Y^{(0)}}{\partial y_{0i}} = 0$$

for all y and y_0 and therefore $M_Y^{(0)}$ is constant. Because $p_Y \to \phi(y)$ as $\Delta \to 0$ and

$$\lim_{\Delta \to 0} \left[l_Y^{(K)}(t, y \mid t_0, y_0) + \frac{m}{2} \ln(2\pi\Delta) + \frac{1}{2\Delta} \sum_{i=1}^m (y_i - y_{0i})^2 \right] = C_Y^{(0)}(t, y \mid t_0, y_0),$$

$M_Y^{(0)} = 0$ so that the limiting density integrates to one rather than $\exp\left(M_Y^{(0)}\right)$ as Δ goes to zero.

Next, for $k = 0$,

$$f_Y^{(0)}(t, y|t_0, y_0) = C_Y^{(1)}(t, y|t_0, y_0) + \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial C_Y^{(1)}(t, y \mid t_0, y_0)}{\partial y_i} - G_Y^{(1)}(t, y \mid t_0, y_0),$$

where $G_Y^{(1)}(t, y \mid t_0, y_0)$ depends on $C_Y^{(0)}(t, y|t_0, y_0)$ and $\mu_Y(t, y)$ as can be seen in (20). The partial differential equation $f_Y^{(0)}(t, y|t_0, y_0) = 0$ of $C_Y^{(1)}(t, y \mid t_0, y_0)$ can be similarly solved as follows:

$$C_Y^{(1)}(t, y \mid t_0, y_0) = \int_0^1 G_Y^{(1)}(t, y_0 + u(y - y_0) \mid t_0, y_0)\, du + \sum_{i,j=1, j\neq i}^m \alpha_{ij}^{(1)} \frac{y_i - y_{0i}}{(y_j - y_{0j})^2} + M_Y^{(1)}.$$

Again the integration constants $\alpha_{ij}^{(1)}$ are zero because $C_Y^{(1)}(t, y \mid t_0, y_0)$ is finite when passing through the axes $y_j = y_{0j}$ for all $j = 1, \cdots, m$. Using the fact that $C_Y^{(1)}(t, y \mid t_0, y_0)$ must satisfy $b_Y^{(0)}(t, y|t_0, y_0) = 0$ too, we have

$$M_Y^{(1)} - \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial M_Y^{(1)}}{\partial y_{0i}} = 0. \tag{51}$$

The unique solution $M_Y^{(1)}$ for (51) for all y and y_0 is $M_Y^{(1)} = 0$, which gives the coefficient $C_Y^{(1)}(t, y \mid t_0, y_0)$.

In general, for all $k \geq 1$

$$f_Y^{(k-1)}(t, y|t_0, y_0) = C_Y^{(k)}(t, y|t_0, y_0) + \frac{1}{k} \sum_{i=1}^m (y_i - y_{0i}) \frac{\partial C_Y^{(k)}(t, y \mid t_0, y_0)}{\partial y_i} - G_Y^{(k)}(t, y \mid t_0, y_0)$$

where $G_Y^{(k)}(t, y \mid t_0, y_0)$ is given in (21) and depends on the previously determined coefficients $C_Y^{(0)}(t, y \mid t_0, y_0)$, $\cdots$, $C_Y^{(k-1)}(t, y \mid t_0, y_0)$ and $\mu_Y(t, y)$. If we solve the differential equation, $f_Y^{(k-1)}(t, y|t_0, y_0) = 0$ for $C_Y^{(k)}(t, y \mid t_0, y_0)$ we obtain

$$C_Y^{(k)}(t, y \mid t_0, y_0) = k \int_0^1 G_Y^{(k)}(t, y_0 + u(y - y_0) \mid t_0, y_0)\, u^{k-1} du + \sum_{i,j=1, j\neq i}^m \alpha_{ij}^{(k)} \frac{y_i - y_{0i}}{(y_j - y_{0j})^{k+1}} + M_Y^{(k)}.$$

The same boundary condition as for $C_Y^{(0)}(t, y \mid t_0, y_0)$ and $C_Y^{(1)}(t, y \mid t_0, y_0)$ and the fact that $b_Y^{(k-1)}(t, y|t_0, y_0) = 0$ make the integration constants $\alpha_{ij}^{(k)} = 0$ for all $i, j = 1, \cdots, m$ $(i \neq j)$ and $M_Y^{(k)} = 0$, respectively, for all $k \geq 1$, which yields the solution (19).

Hence, with those coefficients $C_Y^{(k)}(t, y \mid t_0, y_0)$'s in the theorem, $f_Y^{(k-1)}(t, y|t_0, y_0) = 0$ and $b_Y^{(k-1)}(t, y|t_0, y_0) = 0$ hold for all k, $-1 \leq k \leq K$ and so $F_Y^{(K)}(t, y|t_0, y_0) = O\left(\Delta^K\right)$ and $B_Y^{(K)}(t, y|t_0, y_0) = O\left(\Delta^K\right)$. This proves that (16) solves the Kolmogorov equations to the order Δ^K. Theorem 1 verifies that (16) is actually the right expression for the $K-1$-th order Taylor expansion of log-density of the process Y_t in Δ. ■

Proof of Theorem 2

Proof. As in the proof of Theorem 1, let $F_X^{(K)}(t, x|t_0, x_0)$ and $B_X^{(K)}(t, x|t_0, x_0)$ be the difference between the left- and right-hand sides of (26) and (27), respectively, when $l_X(t, x \mid t_0, x_0)$ is replaced by (25). Rearranging terms of $F_X^{(K)}(t, x|t_0, x_0)$ and $B_X^{(K)}(t, x|t_0, x_0)$ in increasing orders of Δ, we get

$$F_X^{(K)}(t, x|t_0, x_0) = \sum_{k=-2}^{K-1} f_X^{(k)}(t, x|t_0, x_0) \frac{\Delta^k}{k!} + O\left(\Delta^K\right)$$

and

$$B_X^{(K)}(t, x|t_0, x_0) = \sum_{k=-2}^{K-1} b_X^{(k)}(t, x|t_0, x_0) \frac{\Delta^k}{k!} + O\left(\Delta^K\right)$$

with the convention that $(-2)! = (-1)! = 0! = 1$. $f_Y^{(k)}(t, x|t_0, x_0)$'s are given in Theorem 2 and set to zeros to get $F_X^{(K)}(t, x|t_0, x_0) = O\left(\Delta^K\right)$. It is important to not make $D_v(t, x)$, which is known, part of $C_X^{(0)}(t, x \mid t_0, x_0)$ so that $B_X^{(K)}(t, x|t_0, x_0) = O\left(\Delta^K\right)$ can be held. ■

Proof of Proposition 2

Proof. Suppose the diffusion X is reducible then there exists $\gamma(t, x)$ and $C_X^{(k)}(t, x|t_0, x_0) = C_Y^{(k)}[t, \gamma(t, x)|t_0, \gamma(t_0, x_0)]$. By construction, the coefficients of $C_X^{(j_k, k)}(t, x \mid t_0, x_0)$ acquired from the irreducible method are the coefficients of direct Taylor-expansion of $C_X^{(k)}[t, \gamma(t, x)|t_0, \gamma(t_0, x_0)]$ in (t, x) about (t_0, x_0) at order j_k for each k. ■

Proof of Lemma 1

Proof. Using (34) and the symmetry of $\upsilon(t_0, x_0)$

$$\begin{aligned}
& \sum_{i=1}^{m} \upsilon_{ij}(t_0, x_0) \frac{\partial \widehat{C}_X^{(2,-1)}(t, x \mid t_0, x_0)}{\partial x_i} \\
= & \sum_{i=1}^{m} \upsilon_{ij}(t_0, x_0) \left[\sum_{h=1}^{i-1} c_{hi}^{(-1)}(t_0, x_0)(x_h - x_{0h}) + 2c_{i_2}^{(-1)}(t_0, x_0)(x_i - x_{0i}) + \sum_{h=i+1}^{m} c_{ih}^{(-1)}(t_0, x_0)(x_h - x_{0h})\right] \\
= & \sum_{i=1}^{m} \upsilon_{ji}(t_0, x_0) \left[-\sum_{h=1}^{m} \upsilon_{ih}^{-1}(t_0, x_0)(x_h - x_{0h})\right] \\
= & -(x_j - x_{0j})
\end{aligned}$$

for each j. $\blacksquare$

Lemma A1 $\tilde{l}_X^{(\infty)}(t,x \mid t_0,x_0)$ *is free of the indeterminate terms,* $c^{(0)}_{0_{a_0}},\ a_0 \geq 1$.

Proof. Let us Taylor-expand $C_X^{(k)}(t,x|t_0,x_0)$ with respect to x only up to j_k-th order. Then

$$\overline{C}_X^{(j_k,k)}(t,x|x_0) = c^{(k)}(t,x_0) + \sum_{1\leq|a|\leq j_k} c^{(k)}_{1_{a_1}\cdots m_{a_m}}(t,x_0)(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m},$$

where $a=(a_1,a_2,\cdots,a_m)$. It is obvious that Taylor-expanding $\overline{C}_X^{(j_k,k)}(t,x|x_0)$ around t_0 yields $C_X^{(j_k,k)}(t,x|t_0,x_0)$ that includes all indeterminate terms, $c^{(0)}_{0_{a_0}}\Delta^{a_0},\ a_0\geq 1$ in $C_X^{(j_0,0)}(t,x|t_0,x_0)$. The constant term of $\overline{C}_X^{(j_0,0)}(t,x|x_0)$, $c^{(0)}(t,x_0)$, affects all subsequent terms of $c^{(k)}(t,x_0)$ in $\overline{C}_X^{(j_k,k)}(t,x|x_0),\ k\geq 1$ through only $\frac{\partial\overline{C}_X^{(k-1)}(t,x|x_0)}{\partial t}$ in $\overline{G}_X^{(k)}$. Therefore $c^{(k)}(t,x_0),\ k\geq 1$ is equal to $(-1)^k\frac{\partial^k c^{(0)}(t,x_0)}{\partial t^k}$ plus other terms involving t only but not x. Taylor-expanding $c^{(0)}(t,x_0)$ in t around t_0,

$$c^{(0)}(t,x_0) = \sum_{i=1}^{\infty}\frac{\partial^i c^{(0)}(t_0,x_0)}{\partial t^i}\frac{\Delta^i}{i!}$$

because the constant term of $C_X^{(j_0,0)}(t,x|t_0,x_0)$ is zero. Here we abuse notation by using $\frac{\partial^i c^{(0)}(t_0,x_0)}{\partial t^i}$ instead of $\left.\frac{\partial^i c^{(0)}(t,x_0)}{\partial t^i}\right|_{t=t_0}$. Note that $\frac{\partial^i c^{(0)}(t_0,x_0)}{\partial t^i}=c^{(0)}_{0_i}$. Therefore the part of $\overline{C}_X^{(j_k,k)}(t,x\mid x_0)$ influenced by the term $c^{(0)}(t,x_0)$ is

$$\begin{aligned}(-1)^k\frac{\partial^k c^{(0)}(t,x_0)}{\partial t^k} &= (-1)^k\sum_{i=1}^{\infty}\frac{\partial^i c^{(0)}(t_0,x_0)}{\partial t^i}\frac{\partial^k}{\partial t^k}\left[\frac{\Delta^i}{i!}\right]\\ &= (-1)^k\sum_{i\geq k}^{\infty}\frac{\partial^i c^{(0)}(t_0,x_0)}{\partial t^i}\frac{\Delta^{i-k}}{(i-k)!}\end{aligned}$$

for $k\geq 1$.

Because

$$\overline{l}_X^{(K)}(t,x\mid t_0,x_0) = -\frac{m}{2}\ln(2\pi\Delta) - D_\upsilon(x,t;\theta) + \frac{\overline{C}_X^{(j_{-1},-1)}(t,x\mid x_0)}{\Delta} + \sum_{k=0}^{K}\overline{C}_X^{(j_k,k)}(t,x\mid x_0)\frac{\Delta^k}{k!},$$

collecting all of such terms depending on $c^{(0)}(t,x_0)$ in $\overline{C}_X^{(j_k,k)}(t,x\mid x_0),\ k\geq 0$,

$$\begin{aligned}\sum_{k=0}^{\infty}c^{(k)}(t,x\mid x_0;\theta)\frac{\Delta^k}{k!} &= \sum_{k=0}^{\infty}(-1)^k\sum_{i\geq k}^{\infty}\frac{\partial^i c^{(0)}(t_0,x_0)}{\partial t^i}\frac{\Delta^{i-k}}{(i-k)!}\frac{\Delta^k}{k!}\\ &= \sum_{i=0}^{\infty}\frac{\partial^i c^{(0)}(t_0,x_0)}{\partial t^i}\frac{\Delta^i}{i!}\sum_{k=0}^{i}\frac{i!}{(i-k)!k!}(-1)^k\\ &= 0\end{aligned}$$

in $\overline{l}_X^{(\infty)}$. The last equality is because $\sum_{k=0}^{i}\frac{i!}{(i-k)!k!}(-1)^k=(-1+1)^i=0$ for all $i\geq 1$ and $c^{(0)}(t_0,x_0)=0$. This implies that $c^{(0)}_{0_{a_0}},\ a_0\geq 1$, are cancelled out by a_0 consecutive subsequent coefficients $C_X^{(j_k,k)}(t,x|t_0,x_0)$,

$1 \le k \le a_0$. In consequent, as long as we compute $C_X^{(j_k,k)}$ according to (33) $\widetilde{l}_X^{(K)}(t, x \mid t_0, x_0)$ is *free of the indeterminate terms of* $C_X^{(j_0,0)}$. ■

Lemma A2 *The coefficients of* $(x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$, $a = (a_1, \cdots, a_m)$ *and* $|a| \ge 1$ *in* $\widetilde{l}_X^{(\infty)}(t, x \mid t_0, x_0)$ *are not dependent upon the indeterminate terms* of $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$.

Proof. The $(x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ terms in $\widetilde{l}_X^{(\infty)}$ are acquired from the first two coefficients, $\frac{c^{(-1)}_{01_{a_1}2_{a_2}\cdots m_{a_m}} \Delta (x_1-x_{01})^{a_1}(x_2-x_{02})^{a_2}\cdots(x_m-x_{0m})^{a_m}}{\Delta} + c^{(0)}_{1_{a_1}2_{a_2}\cdots m_{a_m}} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m}$. Hence it is sufficient to verify that $c^{(-1)}_{01_{a_1}2_{a_2}\cdots m_{a_m}} + c^{(0)}_{1_{a_1}2_{a_2}\cdots m_{a_m}}$ are the indeterminate terms free for all $|a| \ge 1$. This can be done by mathematical induction on $|a|$ in the similar way to the proof of Lemma A5 using Lemma 1 so every detail is omitted. The differences between the two are the fact that only first two terms matter and none of the L, P, and N terms are involved in this case. ■

Lemma A3 *The coefficients of* $\Delta (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$, $a = (a_1, \cdots, a_m)$ *and* $|a| \ge 1$ *in* $\widetilde{l}_X^{(\infty)}(t, x \mid t_0, x_0)$ *are not dependent upon the indeterminate terms* of $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$.

Proof. The coefficients of $\Delta (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ *of* $\widetilde{l}_X^{(\infty)}$ are attained from the first three coefficients $\frac{c^{(-1)}_{0_2 1_{a_1}2_{a_2}\cdots m_{a_m}} \Delta^2 (x_1-x_{01})^{a_1}(x_2-x_{02})^{a_2}\cdots(x_m-x_{0m})^{a_m}}{\Delta} + c^{(0)}_{01_{a_1}2_{a_2}\cdots m_{a_m}} \Delta (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m} + c^{(1)}_{1_{a_1}2_{a_2}\cdots m_{a_m}} (x_1 - x_{01})^{a_1} (x_2 - x_{02})^{a_2} \cdots (x_m - x_{0m})^{a_m} \Delta$. Given Lemma 1 and Lemma A2, we can show that $c^{(-1)}_{0_2 1_{a_1}2_{a_2}\cdots m_{a_m}} + c^{(0)}_{01_{a_1}2_{a_2}\cdots m_{a_m}} + c^{(1)}_{1_{a_1}2_{a_2}\cdots m_{a_m}}$ are not dependent on any of indeterminate terms of $C_X^{(j_{-1},-1)}(t, x \mid t_0, x_0)$ by induction on $|a|$ in the same way as we prove Lemma A5 below except that $N^{(1,2)}$ term is irrelevant here. In order to save space further proof is not included. ■

Lemma A4 $\sum_{k=i}^{p} \frac{(m+i-1)!}{(m+k)!} (-1)^{k+1-i} \frac{(p-i)!}{(p-k)!} = -\frac{1}{p+m}$ *for all integer* i, $1 \le i \le p$ *and a non-negative integer* m.

Proof. It can be proved by double mathematical induction on i and m. First, consider the case $i = 1$ and $m = 0$.

$$
\begin{aligned}
\sum_{k=1}^{p} \frac{1}{k!} (-1)^k \frac{(p-1)!}{(p-k)!} &= \frac{1}{p} \sum_{k=1}^{p} (-1)^k \frac{p!}{(p-k)!} \frac{1}{k!} \\
&= \frac{1}{p} \left[\sum_{k=0}^{p} (-1)^k \frac{p!}{(p-k)!} \frac{1}{k!} - 1 \right] \\
&= -\frac{1}{p}
\end{aligned}
$$

because $\sum_{k=0}^{p} (-1)^k \frac{p!}{(p-k)!} \frac{1}{k!} = (-1+1)^p = 0$. Suppose that $\sum_{k=h}^{p} \frac{(h-1)!}{k!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} = -\frac{1}{p}$ for $i =$

$h,\ 1 \le h < p$ and $m = 0$. Then, for $i = h + 1$ and $m = 0$,

$$\begin{aligned}
\sum_{k=h+1}^{p} \frac{h!}{k!} (-1)^{k-h} \frac{(p-h-1)!}{(p-k)!} &= \sum_{k=h}^{p} \frac{h!}{k!} (-1)^{k-h} \frac{(p-h-1)!}{(p-k)!} - \frac{1}{(p-h)} \\
&= -\frac{h}{(p-h)} \sum_{k=h}^{p} \frac{(h-1)!}{k!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} - \frac{1}{(p-h)} \\
&= \frac{h}{(p-h)\, p} - \frac{1}{(p-h)} \\
&= -\frac{1}{p}.
\end{aligned}$$

Now assume that $\sum_{k=h}^{p} \frac{(l+h-1)!}{(l+k)!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} = -\frac{1}{p+l}$ for $i = h,\ 1 \le h < p$ and for $m = l \ge 0$. The last step is to show that it holds for $i = h$ and $m = l + 1$. If we let $k + 1 = j$,

$$\begin{aligned}
\sum_{k=h}^{p} \frac{(l+h)!}{(l+1+k)!} (-1)^{k+1-h} \frac{(p-h)!}{(p-k)!} &= \sum_{j=h+1}^{p+1} \frac{(l+h)!}{(l+j)!} (-1)^{j-h} \frac{(p-h)!}{(p+1-j)!} \\
&= \frac{-(l+h)}{(p+1-h)} \sum_{j=h+1}^{p+1} \frac{(l+h-1)!}{(l+j)!} (-1)^{j+1-h} \frac{(p+1-h)!}{(p+1-j)!} \\
&= \frac{-(l+h)}{(p+1-h)} \left[\sum_{j=h}^{p+1} \frac{(l+h-1)!}{(l+j)!} (-1)^{j+1-h} \frac{(p+1-h)!}{(p+1-j)!} + \frac{1}{(l+h)} \right] \\
&= \frac{-(l+h)}{(p+1-h)} \left[-\frac{1}{(p+1+l)} + \frac{1}{(l+h)} \right] \\
&= -\frac{1}{(p+l+1)}.
\end{aligned}$$

■

Lemma A5 *The coefficients of* $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ *,* $a_0 \ge 2$ *and* $a = (a_1, \cdots, a_m)$*,* $|a| \ge 1$ *in* $\widetilde{l}_X^{(\infty)}(t, x \mid t_0, x_0)$ *are not dependent upon the indeterminate terms* of $C_X^{(j_{-1}, -1)}(t, x \mid t_0, x_0)$.

Proof. Notice that the coefficients of $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in $\widetilde{l}_X^{(\infty)}$ are determined by the first $a_0 + 2$ coefficients. Thus, it suffices to show that

$$\sum_{k=-1}^{a_0} c^{(k)}_{0_{a_0-k} 1_{a_1} \cdots m_{a_m}} \frac{1}{k!} = R_{0_{a_0+1} 1_{a_1} \cdots m_{a_m}} \tag{52}$$

for all $a_0 \ge 2$ *and* $a = (a_1, \cdots, a_m)$*,* $|a| \ge 1$. $R_{0_{a_0+1} 1_{a_1} \cdots m_{a_m}}$ represents the coefficient of $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in $\widetilde{l}_X^{(\infty)}$, which is free from the indeterminate terms. The subindices of R that coincides with those of $c^{(-1)}$ tell us whose coefficient it is. In this paper, R without any indices denotes a term that does not depend on the indeterminate terms. Because the same pattern of $G_X^{(k)}$ repeats from $k = 2$ on, this can be proved by double mathematical induction on a_0 and $|a|$. First, the $a_0 = 2$ and $|a| \ge 1$ case of (52) can be verified by induction on $|a|$ and using Lemma 1, Lemma A2, Lemma A3 and Lemma A4 similarly to the last step of double mathematical induction described below and the explanations are left out here.

Now, assume that (52) holds for $a_0 = p \geq 2$ and $|a| \geq 1$, that is $\sum_{k=-1}^{p-1} c_{0_{p-k-1}1_{a_1}\cdots m_{a_m}}^{(k)} \frac{1}{k!} = R_{0_p 1_{a_1}\cdots m_{a_m}}$. Then we can justify that (52) is also valid for $a_0 = p+1$ and $|a| \geq 1$ by induction on $|a|$. Using Lemma 1, Lemma A2, Lemma A3, Lemma A4 and the assumption $\sum_{k=-1}^{p-1} c_{0_{p-k-1}1_{a_1}\cdots m_{a_m}}^{(k)} \frac{1}{k!} = R_{0_p 1_{a_1}\cdots m_{a_m}}$, the $a_0 = p+1$ and $|a| = 1$ case can be attested by showing that the coefficient of $\Delta^p (x_1 - x_{01})$ in $\widetilde{l}_X^{(\infty)} (t, x \mid t_0, x_0)$ does not suffer from the indeterminacy problem without loss of generality. The rest of this inductive step is very close to the last part of this induction so they are excluded.

Suppose that (52) is satisfied for $a_0 = p+1$ and $1 \leq |a| \leq n$, where $a = (a_1 - 1, a_2, \cdots, a_m,)$. We complete the proof if we can show that (52) also holds for $a_0 = p+1$ and $|a| = n+1$, where $a = (a_1, a_2, \cdots, a_m,)$ without losing generality. Note that to verify the $a_0 = p+1$ and $|a| = n+1$ case we use induction on $|a|$ because the fact that (52) is true for $a_0 = p+1$ and $1 \leq |a| \leq n$ is required in addition to the assumption and the previous lemmas.

Considering the coefficient of $\Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in $\widetilde{f}_X^{(-2)} (t, x|t_0, x_0) = 0$,

$$-2c_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} \Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$$
$$= 2 \sum_{l=1}^{m} \sum_{i=1}^{m} \upsilon_{il} (t_0, x_0) \frac{\partial \widehat{C}_X^{(2,-1)}(t,x|t_0,x_0)}{\partial x_i} \frac{c_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} \Delta^{p+1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}}{\partial x_l}$$
$$+2Q_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} \Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m} + R\Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}, \text{ where}$$
$$Q_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} \Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$$
$$= \frac{1}{2} \sum_{\substack{e_0=1 \\ e_0+b_0+d_0=p+1}}^{p} \sum_{b_0=1}^{p} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=1 \\ e+b+d=a}}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \right.$$
$$\left. \frac{\partial c_{0_{b_0}1_{b_1}\cdots m_{b_m} i}^{(-1)} \Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0}1_{e_1}\cdots m_{e_m} j}^{(-1)} \Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j} \right]$$
$$+ \sum_{\substack{e_0=1 \\ e_0+d_0=p+1 \\ \text{If } e_0=p+1 \text{ then } |b|>1 \text{ or } |d|>0}}^{p+1} \sum_{d_0=0}^{p} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=1 \\ e+b+d=a}}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \right.$$
$$\left. \frac{\partial c_{1_{b_1}\cdots m_{b_m} i}^{(-1)} (x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial c_{0_{e_0}1_{e_1}\cdots m_{e_m} j}^{(-1)} \Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j} \right]. \; Q_{0_{a_0}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)}$$

indicates the rest of indeterminate coefficient of $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in the second term of the right hand side of (28).

In the following proof, $L_{0_{a_0}1_{a_1}\cdots m_{a_m}}^{(k)}$, $P_{0_{a_0}1_{a_1}\cdots m_{a_m}}^{(k)}$ and $N_{0_{a_0}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(k)}$ will be used to denote the coefficients of $\Delta^{a_0} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ which depends on the indeterminate terms of $C_X^{(j_{-1},-1)}$ for different parts of $\widetilde{f}_X^{(k-1)}$, $k \geq 0$.

Because of Lemma 1, it is reduced to $(|a| - 1) c_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} \Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m} = Q_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} \Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m} + R\Delta^{p+1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$. Therefore

$$Q_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} = (|a| - 1) c_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}^{(-1)} + R. \tag{53}$$

Next, find the term $\Delta^p (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in $\widetilde{f}_X^{(-1)} (t, x|t_0, x_0) = 0$. Then,

$$-\sum_{l=1}^{m}\sum_{i=1}^{m}\upsilon_{il}\left(t_0,x_0\right)\frac{\partial\widehat{C}_X^{(2,-1)}(t,x|t_0,x_0)}{\partial x_i}\frac{\partial c^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}\Delta^p(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}}{\partial x_1}$$
$$-L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}=-\frac{\partial c^{(-1)}_{0_{p+1}1_{a_1}\cdots m_{a_m}}\Delta^{p+1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}}{\partial t}$$
$$+P^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}+R\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}.$$

From the first term of (29) we get $L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}=$

$$\sum_{\substack{e_0=0\\e_0+b_0+d_0=p\\\text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p}\sum_{b_0=0}^{p}\sum_{d_0=0}^{p}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(0)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

and the $\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$ term from $\widetilde{G}_X^{(0,1)}$ is $P^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$

$$=-\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\mu_i(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\right]$$
$$+\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\left[\frac{\partial\upsilon_{ij}(t_0,x_0)}{\partial x_i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$-\sum_{\substack{e_0=1\\e_0+b_0+d_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial^{d_0+d_1+\cdots+d_m}\left[\frac{\partial D_\upsilon(t_0,x_0)}{\partial x_j}\right]\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\right]$$
$$+\frac{1}{2}\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}ij}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})(x_j-x_{0j})}{\partial x_i\partial x_j}\right].$$

Due to Lemma 1, it can be simplified as $|a|\,c^{(0)}_{0_p1_{i_1}2_{i_2}\cdots m_{i_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$
$-L^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$
$=-\left(p+1\right)c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$
$+P^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}+R\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$ or

$$c^{(0)}_{0_p1_{i_1}2_{i_2}\cdots m_{i_m}}=\frac{1}{|a|}\left[-\left(p+1\right)c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}+L^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}\right]+R. \tag{54}$$

Looking at the term $\Delta^{p-1}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$ in $\widetilde{f}_X^{(0)}\left(t,x|t_0,x_0\right)=0$,

$$c^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}\Delta^{p-1}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$$
$$-\sum_{l=1}^{m}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}\left(t_0,x_0\right)\frac{\partial C_X^{(-1,2)}(t,x|t_0,x_0)}{\partial x_i}\frac{\partial c^{(1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}}{\partial x_l}$$
$$-L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}\Delta^{p-1}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$$

$= -\frac{\partial c^{(0)}_{0_p 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^p (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}}{\partial t} + P^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$
$+ N^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m} + R\Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$, where
$L^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m} =$
$\sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1 \\ \text{If } e_0=p-1 \text{ then } |b|>1 \text{ or } |d|>0}}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0) \Delta^{d_0} (x_1-x_{01})^{d_1}}{\partial t^{d_0} \partial x_1^{d_1} \cdots \partial x_m^{d_m}} \times \right.$
$\left. \frac{\partial c^{(-1)}_{0_{b_0} 1_{b_1} \cdots m_{b_m} i} \Delta^{b_0} (x_1-x_{01})^{b_1} \cdots (x_m-x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c^{(1)}_{0_{e_0} 1_{e_1} \cdots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \cdots (x_m-x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]$, which is from the second term of $\widetilde{f}^{(0)}_X$.

Similarly to $P^{(0)}_{0_p 1_{a_1} 2_{a_2} \cdots m_{a_m}}$, the coefficient of $\Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$ in $\widetilde{G}^{(1,1)}_X$ is
$P^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$
$= - \sum_{\substack{e_0=0 \\ e_0+b_0=p-1}}^{p-1} \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=0 \\ e+b=a}}^{n+1} \sum_{i=1}^{m} \left[\frac{\partial^{b_0+b_1+\cdots+b_m} \mu_i(t_0,x_0) \Delta^{b_0} (x_1-x_{01})^{b_1} \cdots (x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \cdots \partial x_m^{b_m}} \times \right.$
$\left. \frac{\partial c^{(0)}_{0_{e_0} 1_{e_1} \cdots m_{e_m} i} \Delta^{e_0} (x_1-x_{01})^{e_1} \cdots (x_m-x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \right]$
$+ \sum_{\substack{e_0=0 \\ e_0+b_0=p-1}}^{p-1} \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=0 \\ e+b=a}}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{b_0+b_1+\cdots+b_m} \left[\frac{\partial \upsilon_{ij}(t_0,x_0)}{\partial x_i} \right] \Delta^{b_0} (x_1-x_{01})^{b_1} \cdots (x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \cdots \partial x_m^{b_m}} \times \right.$
$\left. \frac{\partial c^{(0)}_{0_{e_0} 1_{e_1} \cdots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \cdots (x_m-x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]$
$- \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{b_0+b_1+\cdots+b_m} \upsilon_{ij}(t_0,x_0) \Delta^{b_0} (x_1-x_{01})^{b_1} \cdots (x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \cdots \partial x_m^{b_m}} \times \right.$
$\left. \frac{\partial c^{(0)}_{0_{e_0} 1_{e_1} \cdots m_{e_m} i} \Delta^{e_0} (x_1-x_{01})^{e_1} \cdots (x_m-x_{0m})^{e_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial^{d_0+d_1+\cdots+d_m} \left[\frac{\partial D_\upsilon(t_0,x_0)}{\partial x_j} \right] \Delta^{d_0} (x_1-x_{01})^{d_1} \cdots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \cdots \partial x_m^{d_m}} \right]$
$+ \frac{1}{2} \sum_{\substack{e_0=0 \\ e_0+b_0=p-1}}^{p-1} \sum_{b_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=0 \\ e+b=a}}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{b_0+b_1+\cdots+b_m} \upsilon_{ij}(t_0,x_0) \Delta^{b_0} (x_1-x_{01})^{b_1} \cdots (x_m-x_{0m})^{b_m}}{\partial t^{b_0} \partial x_1^{b_1} \cdots \partial x_m^{b_m}} \times \right.$
$\left. \frac{\partial c^{(0)}_{0_{e_0} 1_{e_1} \cdots m_{e_m} ij} \Delta^{e_0} (x_1-x_{01})^{e_1} \cdots (x_m-x_{0m})^{e_m} (x_i - x_{0i})(x_j - x_{0j})}{\partial x_i \partial x_j} \right]$ and from $\widetilde{G}^{(1,2)}_X$ we obtain
$N^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$
$= \frac{1}{2} \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p-1}}^{p-1} \sum_{b_0=0}^{p-1} \sum_{d_0=0}^{p-1} \sum_{|e|=0}^{n+1} \sum_{\substack{|b|=0 \\ e+b+d=a}}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0) \Delta^{d_0} (x_1-x_{01})^{d_1} \cdots (x_m-x_{0m})^{d_m}}{\partial t^{d_0} \partial x_1^{d_1} \cdots \partial x_m^{d_m}} \times \right.$
$\left. \frac{\partial c^{(0)}_{0_{b_0} 1_{b_1} \cdots m_{b_m} i} \Delta^{b_0} (x_1-x_{01})^{b_1} \cdots (x_m-x_{0m})^{b_m} (x_i - x_{0i})}{\partial x_i} \frac{\partial c^{(0)}_{0_{e_0} 1_{e_1} \cdots m_{e_m} j} \Delta^{e_0} (x_1-x_{01})^{e_1} \cdots (x_m-x_{0m})^{e_m} (x_j - x_{0j})}{\partial x_j} \right]$. Notice that $\widetilde{G}^{(1,3)}_X$ is independent of the indeterminate terms.

By Lemma 1, it can be written as
$(|a|+1)\, c^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m} - L^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$
$= -p c^{(0)}_{0_p 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m} + P^{(1)}_{0_{p-1} 1_{a_1} 2_{a_2} \cdots m_{a_m}} \Delta^{p-1} (x_1 - x_{01})^{a_1} \cdots (x_m - x_{0m})^{a_m}$

$+N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}\Delta^{p-1}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}+R\Delta^{p-1}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$. Hence

$$c^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}=\frac{1}{(|a|+1)}\left[-pc^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}+L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}\right]+R. \tag{55}$$

Now, turn to the term $\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$ in $\widetilde{f}^{(k-1)}_X\left(t,x|t_0,x_0\right)=0$ for $2\leq k\leq p$.

$$c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}-$$
$$\frac{1}{k}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}\left(t_0,x_0\right)\frac{\partial C^{(-1,2)}_X(t,x|t_0,x_0)}{\partial x_i}\frac{\partial c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}}{\partial x_1}-$$
$$L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$$
$$=-\frac{\partial c^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}\Delta^{p-k+1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}}{\partial t}+P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}+$$
$N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}+R\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$. Similarly to before, we get $L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}$ from the second part of $\widetilde{f}^{(k-1)}_X$, $P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}$ from $\widetilde{G}^{(k,1)}_X$, and $N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}$ from $\widetilde{G}^{(k,2)}_X$. Here

$$L^{(k)}_{0_{p-k}1}\Delta^{p-k}\left(x_1-x_{01}\right)=$$
$$\frac{1}{k}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k\\ \text{If } e_0=p-k \text{ then } |b|>1 \text{ or } |d|>0}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right],$$

$$P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}=$$
$$\sum_{\substack{e_0=0\\ e_0+b_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b=a}}^{n+1}\sum_{i=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\mu_i(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\right]$$
$$+\sum_{\substack{e_0=0\\ e_0+b_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b=a}}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\left[\frac{\partial\upsilon_{ij}(t_0,x_0)}{\partial x_i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$-\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial^{d_0+d_1+\cdots+d_m}\left[\frac{\partial D\upsilon(t_0,x_0)}{\partial x_j}\right]\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\right]$$
$$+\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b=a}}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}ij}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})(x_j-x_{0j})}{\partial x_i\partial x_j}\right] \text{ and}$$
$$N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}=$$

$$\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial c^{(0)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\right\}.$$

Then

$\left[1+\frac{1}{k}|a|\right]c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}-$
$L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}=$
$-(p-k+1)\,c^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+$
$P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+$
$R\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$ by using Lemma 1. Thus

$$c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}=\frac{k}{k+|a|}\left[L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}-(p-k+1)\,c^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}+P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\right]+R \tag{56}$$

for $2\le k\le p$.

The fact that $\sum_{k=-1}^{r}c^{(k)}_{0_{r-k}1_{a_1}\cdots m_{a_m}}\frac{1}{k!}=R_{0_{r+1}1_{a_1}\cdots m_{a_m}}$, for $0\le r\le p-1$ and $|a|\ge 1$ and the assumption $\sum_{k=-1}^{p}c^{(k)}_{0_{p-k}1_{a_1-1}\cdots m_{a_m}}\frac{1}{k!}=R_{0_{p+1}1_{a_1-1}\cdots m_{a_m}}$, where $p\ge 3$ and $1\le|a|\le n$ play important roles in deriving the following results.

$$P^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}\Delta^p(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$$
$$=-\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\mu_i(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial\left[\sum_{k=0}^{e_0-1}\left(-c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}+R_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\right]$$
$$+\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\left[\frac{\partial v_{ij}(t_0,x_0)}{\partial x_i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\left.\frac{\partial\left[\sum_{k=0}^{e_0-1}\left(-c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}j}\frac{1}{k!}+R_{0_{e_0-k}1_{e_1}\cdots m_{e_m}j}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$-\sum_{\substack{e_0=1\\e_0+b_0+d_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}v_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$
$$\frac{\partial\left[\sum_{k=0}^{e_0-1}\left(-c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}+R_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\times$$

$$\left.\frac{\partial^{d_0+d_1+\cdots+d_m}\left[\frac{\partial D_v(t_0,x_0)}{\partial x_j}\right]\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$

$$\left.\frac{\left[\sum_{k=0}^{e_0-1}\left(-c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}ij}\frac{1}{k!}+R_{0_{e_0-k}1_{e_1}\cdots m_{e_m}ij}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})(x_j-x_{0j})}{\partial x_i\partial x_j}\right]$$

$$=\sum_{k=0}^{p-1}\sum_{\substack{e_0=1+k\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\mu_i(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$

$$\left.\frac{\partial\left[\left(c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\right]$$

$$-\sum_{k=0}^{p-1}\sum_{\substack{e_0=1+k\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\left[\frac{\partial\upsilon_{ij}(t_0,x_0)}{\partial x_i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$

$$\left.\frac{\partial\left[\left(c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}j}\frac{1}{k!}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\sum_{k=0}^{p-1}\sum_{\substack{e_0=1+k\\e_0+b_0+d_0=p}}^{p}\sum_{b_0=0}^{p-k-1}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$

$$\frac{\partial\left[\left(c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})}{\partial x_i}\times$$

$$\left.\frac{\partial^{d_0+d_1+\cdots+d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\left[\frac{\partial D_v(t_0,x_0)}{\partial x_j}\right]\Delta^{d_0}\left(x_1-x_{01}\right)^{d_1}\cdots\left(x_m-x_{0m}\right)^{d_m}\right]$$

$$-\frac{1}{2}\sum_{k=0}^{p-1}\sum_{\substack{e_0=1+k\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{b_0+b_1+\cdots+b_m}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}}{\partial t^{b_0}\partial x_1^{b_1}\cdots\partial x_m^{b_m}}\times\right.$$

$$\left.\frac{\left[\left(c^{(k)}_{0_{e_0-(k+1)}1_{e_1}\cdots m_{e_m}ij}\frac{1}{k!}\right)\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_i-x_{0i})(x_j-x_{0j})}{\partial x_i\partial x_j}\right]+R\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$$

$=-\sum_{k=1}^{p}P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}\frac{1}{(k-1)!}+R\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$. Therefore

$$P^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}=-\sum_{k=1}^{p}P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{(k-1)!}+R. \tag{57}$$

For $2\leq k\leq p-1$, $N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}=$

$$\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial c^{(0)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\Bigg\}$$

$$=\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\frac{\partial\left[-c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}-\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}+R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\times$$

$$\left.\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\right\}$$

$$=-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\right\}$$

$$=-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k+1}}^{p-k}\sum_{b_0=1}^{p-k+1}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\right\}$$

$= N^{(k,1)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+N^{(k,2)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+$ $R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$, where $N^{(k,1)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$=-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k+1}}^{p-k}\sum_{b_0=1}^{p-k+1}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right],$$

$N^{(k,2)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$=-\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\right\}$$

and

$R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\Delta^{p-k}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$=\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right].$$

So,

$$N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}=N^{(k,1)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+N^{(k,2)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\tag{58}$$

for $2\le k\le p-1$.

When $k=p$,

$$N^{(p)}_{1_{a_1}\cdots m_{a_m}} = N^{(p,1)}_{1_{a_1}\cdots m_{a_m}} + N^{(p,2)}_{1_{a_1}\cdots m_{a_m}} + R^{N^{(p)}_{1_{a_1}\cdots m_{a_m}}} \tag{59}$$

but $N^{(p,1)}_{1_{a_1}\cdots m_{a_m}}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$= -\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(-1)}_{01_{b_1}\cdots m_{b_m}i}\right](x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(p-1)}_{1_{e_1}\cdots m_{e_m}j}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right],$$

$N^{(p,2)}_{1_{a_1}\cdots m_{a_m}}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$= \frac{1}{2}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\left[\sum_{h=1}^{p-2}\binom{p-1}{h}\frac{\partial c^{(h)}_{1_{b_1}\cdots m_{b_m}i}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(p-1-h)}_{1_{e_1}\cdots m_{e_m}j}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\right\}$$ and

$R^{N^{(p)}_{1_{a_1}\cdots m_{a_m}}}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$= \sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[R_{01_{b_1}\cdots m_{b_m}i}\right](x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(p-1)}_{1_{e_1}\cdots m_{e_m}j}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right].$$

If $k=1$,

$N^{(1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m} =$

$$\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(0)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(0)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$= \frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\frac{\partial\left[-c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}-\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}+R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\times$$
$$\left.\frac{\partial\left[-c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}-\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}+R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$= \frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}j}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\times$$

$$\left.\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}j}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}j}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+R\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$$

$$=\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\times$$

$$\left.\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$+R\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$
$=N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+N^{(1,2)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+$
$R^{N^{(1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}+R\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$, i.e.

$$N^{(1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}=N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}+N^{(1,2)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}+R^{N^{(1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}}+R. \tag{60}$$

Here $N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$=\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$=\frac{1}{2}\sum_{\substack{e_0=1\\e_0+b_0+d_0=p+1}}^{p}\sum_{b_0=1}^{p}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0-1}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right] \text{ and}$$

$R^{N^{(1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$

$$=-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{\substack{|e|=0\\e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right].$$

Looking at $N^{(1,2)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}$ term in more detail,

$$N^{(1,2)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$$
$$=\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$+\frac{1}{2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\times$$
$$\left.\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$=\sum_{\substack{e_0=1\\ e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-2}\sum_{d_0=0}^{p-2}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$+\frac{1}{2}\sum_{\substack{e_0=1\\ e_0+b_0+d_0=p-1}}^{p-2}\sum_{b_0=1}^{p-2}\sum_{d_0=0}^{p-3}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\times$$
$$\left.\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$=\sum_{k=1}^{p-1}\sum_{e_0=k}^{p-1}\sum_{\substack{b_0=0\\ e_0+b_0+d_0=p-1}}^{p-k-1}\sum_{d_0=0}^{p-k-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$
$$+\frac{1}{2}\sum_{k=1}^{p-2}\sum_{e_0=k}^{p-2}\sum_{\substack{b_0=1\\ e_0+b_0+d_0=p-1}}^{p-k-1}\sum_{d_0=0}^{p-k-2}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$
=\sum_{k=1}^{p-1}\sum_{\substack{e_0=k\\ e_0+b_0+d_0=p}}^{p-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times
$$

$$
\frac{\partial\left[c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]
$$

$$
+\frac{1}{2}\sum_{k=1}^{p-2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k-1}}^{p-k-2}\sum_{b_0=1}^{p-k-1}\sum_{d_0=0}^{p-k-2}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times
$$

$$
\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[c^{(k)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0+k}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]
$$

$$
=\sum_{k=1}^{p-1}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times
$$

$$
\frac{\partial\left[c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[c^{(k)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0+k}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]
$$

$$
+\frac{1}{2}\sum_{k=1}^{p-2}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k-1}}^{p-k-2}\sum_{b_0=1}^{p-k-1}\sum_{d_0=0}^{p-k-2}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times
$$

$$
\frac{\partial\left[\sum_{l=1}^{b_0}c^{(l)}_{0_{b_0-l}1_{b_1}\cdots m_{b_m}i}\frac{1}{l!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial\left[c^{(k)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0+k}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]
$$

$=-\sum\limits_{k=2}^{p}N^{(k,1)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{(k-1)!}\Delta^{p-1}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$. Consequently,

$$
N^{(1,2)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}=-\sum_{k=2}^{p}N^{(k,1)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{(k-1)!}. \tag{61}
$$

Let us show that

$$
\sum_{k=2}^{p}N^{(k,2)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{(k-1)!}=0. \tag{62}
$$

$$
\sum_{k=2}^{p}N^{(k,2)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}\frac{\Delta^k}{(k-1)!}
$$

$$
=-\frac{1}{2}\sum_{k=2}^{p-1}\frac{\Delta^k}{(k-1)!}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times
$$

$$
\frac{\partial\left[\sum_{h=1}^{b_0}c^{(h)}_{0_{b_0-h}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]
$$

$$
+\frac{1}{2}\sum_{k=3}^{p}\frac{\Delta^k}{(k-1)!}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg\{\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times
$$

$$
\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\Bigg\}=
$$

0 because

$$
-\frac{1}{2}\sum_{k=2}^{p-1}\frac{\Delta^k}{(k-1)!}\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{\substack{|e|=0\\ e+b+d=a}}^{n+1}\sum_{|b|=0}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times
$$

$$\left.\frac{\partial\left[\sum_{h=1}^{b_0} c^{(h)}_{0_{b_0-h}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$=-\frac{1}{2}\sum_{k=3}^{p}\frac{\Delta^{k-1}}{(k-2)!}\sum_{e_0=0}^{p-k}\sum_{\substack{b_0=1\\ e_0+b_0+d_0=p-k+1}}^{p-k+1}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[\sum_{h=1}^{b_0} c^{(h)}_{0_{b_0-h}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-2)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$=-\frac{1}{2}\sum_{k=3}^{p}\sum_{h=1}^{p-k+1}\sum_{e_0=0}^{p-k-h+1}\sum_{\substack{b_0=h\\ e_0+b_0+d_0=p-k+1}}^{p-k+1}\sum_{d_0=0}^{p-k-h+1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(h)}_{0_{b_0-h}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0+k-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-2)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\frac{1}{(k-2)!}$$

$$=-\frac{1}{2}\sum_{k=3}^{p}\sum_{h=1}^{p-k+1}\sum_{e_0=0}^{p-k-h+1}\sum_{\substack{b_0=0\\ e_0+b_0+d_0=p-k-h+1}}^{p-k-h+1}\sum_{d_0=0}^{p-k-h+1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0+k+h-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-2)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\frac{1}{(k-2)!}$$

$$=-\frac{1}{2}\sum_{h=1}^{p-2}\sum_{k=3}^{p-h+1}\sum_{e_0=0}^{p-k-h+1}\sum_{\substack{b_0=0\\ e_0+b_0+d_0=p-k-h+1}}^{p-k-h+1}\sum_{d_0=0}^{p-k-h+1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0+k+h-1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-2)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\frac{1}{(k-2)!}$$

$$=-\frac{1}{2}\sum_{h=1}^{p-2}\sum_{k=h+2}^{p}\sum_{e_0=0}^{p-k}\sum_{\substack{b_0=0\\ e_0+b_0+d_0=p-k}}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0+k}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-h-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\frac{1}{(k-h-1)!}$$

$$=-\frac{1}{2}\sum_{h=1}^{p-2}\sum_{k=h+2}^{p}\sum_{e_0=0}^{p-k}\sum_{\substack{b_0=0\\ e_0+b_0+d_0=p-k}}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0+k}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(k-h-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\frac{1}{(k-h-1)!}$$

$$=-\frac{1}{2}\sum_{s=1}^{p-2}\sum_{r=s+2}^{p}\sum_{e_0=0}^{p-r}\sum_{\substack{b_0=0\\ e_0+b_0+d_0=p-r}}^{p-r}\sum_{d_0=0}^{p-r}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$
$$\left.\frac{\partial\left[c^{(s)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\frac{1}{h!}\right]\Delta^{b_0+r}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\,\frac{\partial c^{(r-s-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\frac{1}{(r-s-1)!}$$

and

$$\frac{1}{2}\sum_{k=3}^{p}\frac{\Delta^{k}}{(k-1)!}\sum_{e_0=0}^{p-k}\sum_{\substack{b_0=0\\ e_0+b_0+d_0=p-k}}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left\{\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times\right.$$

$$\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\Bigg\}$$

$$=\frac{1}{2}\sum_{k=3}^{p}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg\{\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\left[\sum_{h=1}^{k-2}\frac{1}{(k-1-h)!h!}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0+k}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\Bigg\}$$

$$=\frac{1}{2}\sum_{h=1}^{p-2}\sum_{k=h+2}^{p}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg\{\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\left[\frac{1}{h!}\frac{\partial c^{(h)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0+k}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{1}{(k-1-h)!}\frac{\partial c^{(k-1-h)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]\Bigg\}.$$

Let us see the $L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}$ term.

$$L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}\Delta^p(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}=$$

$$\sum_{\substack{e_0=0\\e_0+b_0+d_0=p\\\text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p}\sum_{b_0=0}^{p}\sum_{d_0=0}^{p}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(0)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]$$

$$=\sum_{\substack{e_0=0\\e_0+b_0+d_0=p\\\text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p}\sum_{b_0=0}^{p}\sum_{d_0=0}^{p}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\times$$

$$\frac{\partial\left[-c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}-\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}+R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]$$

$$=-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p\\\text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p}\sum_{b_0=0}^{p}\sum_{d_0=0}^{p}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(-1)}_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]$$

$$-\sum_{\substack{e_0=1\\e_0+b_0+d_0=p\\\text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]$$

$$+\sum_{\substack{e_0=0\\e_0+b_0+d_0=p\\\text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p}\sum_{b_0=0}^{p}\sum_{d_0=0}^{p}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]$$

$$= -\sum_{\substack{e_0=1 \\ e_0+b_0+d_0=p+1 \\ \text{If } e_0=p+1 \text{ then } |b|>1 \text{ or } |d|>0}}^{p+1} \sum_{b_0=0}^{p} \sum_{d_0=0}^{p} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial\left[c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0-1}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$- \sum_{k=1}^{p} \sum_{\substack{e_0=k \\ e_0+b_0+d_0=p \\ \text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p} \sum_{b_0=0}^{p-k} \sum_{d_0=0}^{p-k} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial\left[c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$+R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$$

$$= -\frac{1}{2} \sum_{\substack{e_0=1 \\ e_0+b_0+d_0=p+1}}^{p} \sum_{b_0=1}^{p} \sum_{d_0=0}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial\left[c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0-1}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\frac{1}{2} \sum_{\substack{e_0=1 \\ e_0+b_0+d_0=p+1}}^{p} \sum_{b_0=1}^{p} \sum_{d_0=0}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial\left[c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0-1}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$- \sum_{\substack{e_0=1 \\ e_0+d_0=p+1 \\ \text{If } e_0=p+1 \text{ then } |b|>1 \text{ or } |d|>0}}^{p+1} \sum_{d_0=0}^{p} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}} \frac{\partial c^{(-1)}_{1_{b_1}\cdots m_{b_m}i}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial\left[c^{(-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0-1}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$-\sum_{k=1}^{p-1} L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}\frac{1}{(k-1)!}$$

$$+R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$$

$$= -Q^{(-1)}_{0_{p+1}1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m} - N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$$

$$-\sum_{k=1}^{p-1} L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}\frac{1}{(k-1)!}+R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m},$$

where $R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}\Delta^p\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$

$$= \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p \\ \text{If } e_0=p \text{ then } |b|>1 \text{ or } |d|>0}}^{p} \sum_{b_0=0}^{p} \sum_{d_0=0}^{p} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i} \frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right]$$

$$= \sum_{\substack{e_0=0 \\ e_0+b_0+d_0=p}}^{p-1} \sum_{b_0=1}^{p} \sum_{d_0=0}^{p-1} \sum_{\substack{|e|=0 \\ e+b+d=a}}^{n+1} \sum_{|b|=0}^{n+1} \sum_{|d|=0}^{n+1} \sum_{i=1}^{m} \sum_{j=1}^{m} \left[\frac{\partial^{d_0+d_1+\cdots+d_m} \upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}} \times \right.$$

$$\frac{\partial c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]+$$

$R\Delta^p(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$.

Accordingly,

$$L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}=-Q^{(-1)}_{0_{p+1}1_{a_1}\cdots m_{a_m}}-N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}-\sum_{k=1}^{p-1}L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{(k-1)!}+R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}+R. \quad (63)$$

Notice that $R^{N^{(1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}}\Delta^{p-1}(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}\frac{\Delta}{1!}$

$$=-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial\left[c^{(-1)}_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0+1}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]$$

$$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]\frac{\Delta}{1!}$$

$$=-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p}}^{p-1}\sum_{b_0=1}^{p}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial\left[c^{(-1)}_{0_{b_0}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[R_{0_{e_0+1}1_{e_1}\cdots m_{e_m}j}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]$$

$$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[\sum_{k=1}^{e_0}c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]\frac{\Delta}{1!}.$$

First two lines of the last equation are equal to $-R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}\Delta^p(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$ and last two lines of the last equation can be written as

$$-\sum_{k=1}^{p-1}\sum_{\substack{e_0=k\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-k-1}\sum_{d_0=0}^{p-k-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(k)}_{0_{e_0-k}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]\frac{\Delta}{1!}\,(k)$$

$$=-\sum_{k=1}^{p-1}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k-1}}^{p-k-1}\sum_{b_0=0}^{p-k-1}\sum_{d_0=0}^{p-k-1}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\Bigg[\frac{\partial^{d_0+d_1+\cdots+d_m}v_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\times$$

$$\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial\left[c^{(k)}_{0_{e_0}1_{e_1}\cdots m_{e_m}i}\frac{1}{k!}\right]\Delta^{e_0+k}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\Bigg]\frac{\Delta}{1!}$$

$=-\sum_{k=2}^{p}R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\frac{1}{(k-1)!}\Delta^p(x_1-x_{01})^{a_1}\cdots(x_m-x_{0m})^{a_m}$ because, for $2\leq k\leq p$,

$R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\Delta^{p-k}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$
$=\sum_{\substack{e_0=0\\ e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{|e|=0}^{n+1}\sum_{\substack{|b|=0\\ e+b+d=a}}^{n+1}\sum_{|d|=0}^{n+1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0+d_1+\cdots+d_m}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}(x_1-x_{01})^{d_1}\cdots(x_m-x_{0m})^{d_m}}{\partial t^{d_0}\partial x_1^{d_1}\cdots\partial x_m^{d_m}}\right.\times$
$\left.\frac{\partial\left[R_{0_{b_0+1}1_{b_1}\cdots m_{b_m}i}\right]\Delta^{b_0}(x_1-x_{01})^{b_1}\cdots(x_m-x_{0m})^{b_m}(x_i-x_{0i})}{\partial x_i}\frac{\partial c^{(k-1)}_{0_{e_0}1_{e_1}\cdots m_{e_m}j}\Delta^{e_0}(x_1-x_{01})^{e_1}\cdots(x_m-x_{0m})^{e_m}(x_j-x_{0j})}{\partial x_j}\right].$

For these reasons, $R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}\Delta^{p}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}=$
$-\sum_{k=1}^{p}R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\frac{1}{(k-1)!}\Delta^{p}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}+R\Delta^{p}\left(x_1-x_{01}\right)^{a_1}\cdots\left(x_m-x_{0m}\right)^{a_m}$ and

$$R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}=-\sum_{k=1}^{p}R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\frac{1}{(k-1)!}+R. \tag{64}$$

Making use of (54), (55), and (56), $c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}$, for $1\le k\le p-1$ and $c^{(p)}_{1_{a_1}\cdots m_{a_m}}$ can be expressed in terms of $c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}$, L, P, and N as follows.

When $1\le k\le p-1$, $c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}$
$=\frac{k!|a|!}{(|a|+k)!}\frac{1}{|a|}\left(-1\right)^{k+1}\frac{(p+1)!}{(p-k)!}c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}$
$+\frac{k!|a|!}{(|a|+k)!}\frac{1}{|a|}\left(-1\right)^{k}\frac{p!}{(p-k)!}\left(L^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}\right)$
$+\frac{k!|a|!}{(|a|+k)!}\left(-1\right)^{k-1}\frac{(p-1)!}{(p-k)!}\left(L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}\right)$
$+\frac{k!(|a|+1)!}{1!(|a|+k)!}\left(-1\right)^{k-2}\frac{(p-2)!}{(p-k)!}\left(L^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+P^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+N^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}\right)$
$+\frac{k!(|a|+2)!}{2!(|a|+k)!}\left(-1\right)^{k-3}\frac{(p-3)!}{(p-k)!}\left(L^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+P^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+N^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}\right)$
$+\cdots+$
$+\frac{k!(|a|+k-2)!}{(k-2)!(|a|+k)!}\left(-1\right)\frac{(p-k+1)!}{(p-k)!}\left(L^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}+P^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}+N^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}\right)$
$+\frac{k!(|a|+k-1)!}{(k-1)!(|a|+k)!}\left(-1\right)^{0}\frac{(p-k)!}{(p-k)!}\left(L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\right)$ and if $k=p$,
$c^{(p)}_{1_{a_1}\cdots m_{a_m}}$
$=\frac{p!|a|!}{(|a|+p)!}\frac{1}{|a|}\left(-1\right)^{p+1}\frac{(p+1)!}{0!}c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}$
$+\frac{p!|a|!}{(|a|+p)!}\frac{1}{|a|}\left(-1\right)^{p}\frac{p!}{0!}\left(L^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}\right)$
$+\frac{p!|a|!}{(|a|+p)!}\left(-1\right)^{p-1}\frac{(p-1)!}{0!}\left(L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}\right)$
$+\frac{p!(|a|+1)!}{1!(|a|+p)!}\left(-1\right)^{p-2}\frac{(p-2)!}{0!}\left(L^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+P^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+N^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}\right)$
$+\frac{p!(|a|+2)!}{2!(|a|+p)!}\left(-1\right)^{p-3}\frac{(p-3)!}{0!}\left(L^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+P^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+N^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}\right)$
$+\cdots+$
$+\frac{p!(|a|+p-2)!}{(p-2)!(|a|+p)!}\left(-1\right)\frac{1!}{0!}\left(L^{(p-1)}_{01_{a_1}\cdots m_{a_m}}+P^{(p-1)}_{01_{a_1}\cdots m_{a_m}}+N^{(p-1)}_{01_{a_1}\cdots m_{a_m}}\right)$
$+\frac{p!(|a|+p-1)!}{(p-1)!(|a|+p)!}\left(P^{(p)}_{1_{a_1}\cdots m_{a_m}}+N^{(p)}_{1_{a_1}\cdots m_{a_m}}\right).$

In $c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}$ and $c^{(p)}_{1_{a_1}\cdots m_{a_m}}$, if we replace $P^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}$ with $-\sum_{k=1}^{p}P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{(k-1)!}$ due to (57), $L^{(0)}_{0_p1_{a_1}2_{a_2}\cdots m_{a_m}}$ with the right-hand side of (63) after putting $\left(|a|-1\right)c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}+R$ in place of $Q^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}$ because of (53), $N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}$ with the right-hand side of (60) after substituting $-\sum_{k=2}^{p}N^{(k,1)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{(k-1)!}$ for $N^{(1,2)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}$ as a result from (61), $N^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}$ with the right-hand side of

(58) for $k=2$ after substituting $N^{(2,2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}$ by $\sum\limits_{l=3}^{p} N^{(l,2)}_{0_{p-l}1_{a_1}\cdots m_{a_m}}\frac{1}{(l-1)!}$ thanks to (62), and $N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}$ with the right-hand side of (58) for $3\le k\le p$, then we get, for $1\le k\le p-1$,

$$
\begin{aligned}
& c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\tfrac{1}{k!} \\
& = \tfrac{|a|!}{(|a|+k)!}\tfrac{1}{|a|}(-1)^{k+1}\tfrac{(p+1)!}{(p-k)!}c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}} \\
& +\tfrac{|a|!}{(|a|+k)!}\tfrac{1}{|a|}(-1)^{k}\tfrac{p!}{(p-k)!}\Bigg[-(|a|-1)\,c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}-N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}-\sum_{k=1}^{p-1}L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\tfrac{1}{(k-1)!}+ \\
& \qquad R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}-\sum_{k=1}^{p}P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\tfrac{1}{(k-1)!}\Bigg] \\
& +\tfrac{|a|!}{(|a|+k)!}(-1)^{k-1}\tfrac{(p-1)!}{(p-k)!}\Bigg[L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+N^{(1,1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}- \\
& \qquad \sum_{i=2}^{p}N^{(i,1)}_{0_{p-i}1_{a_1}2_{a_2}\cdots m_{a_m}}\tfrac{1}{(i-1)!}+R^{N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}}\Bigg] \\
& +\tfrac{(|a|+1)!}{1!(|a|+k)!}(-1)^{k-2}\tfrac{(p-2)!}{(p-k)!}\Bigg[L^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+P^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+N^{(2,1)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}- \\
& \qquad \sum_{l=3}^{p}N^{(l,2)}_{0_{p-l}1_{a_1}\cdots m_{a_m}}\tfrac{1}{(l-1)!}+R^{N^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}}\Bigg] \\
& +\tfrac{(|a|+2)!}{2!(|a|+k)!}(-1)^{k-3}\tfrac{(p-3)!}{(p-k)!}\Bigg[L^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+P^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+N^{(3,1)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+ \\
& N^{(3,2)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+R^{N^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}}\Bigg]+\cdots \\
& +\tfrac{(|a|+k-2)!}{(k-2)!(|a|+k)!}(-1)\tfrac{(p-k+1)!}{(p-k)!}\Bigg[L^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}+P^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}+N^{(k-1,1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}+ \\
& N^{(k-1,2)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}+R^{N^{(k-1)}_{0_{p-k+1}1_{a_1}\cdots m_{a_m}}}\Bigg] \\
& +\tfrac{(|a|+k-1)!}{(k-1)!(|a|+k)!}(-1)^{0}\tfrac{(p-k)!}{(p-k)!}\Bigg[L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+N^{(k,1)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+ \\
& N^{(k,2)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}+R^{N^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}}\Bigg] \text{ and for } k=p,
\end{aligned}
$$

$$
\begin{aligned}
& c^{(p)}_{1_{a_1}\cdots m_{a_m}}\tfrac{1}{p!}=\tfrac{|a|!}{(|a|+p)!}\tfrac{1}{|a|}(-1)^{p+1}\tfrac{(p+1)!}{0!}c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}} \\
& +\tfrac{|a|!}{(|a|+p)!}\tfrac{1}{|a|}(-1)^{p}\tfrac{p!}{0!}\Bigg[-(|a|-1)\,c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}-N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}}-\sum_{k=1}^{p-1}L^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\tfrac{1}{(k-1)!}+ \\
& R^{L^{(0)}_{0_p1_{a_1}\cdots m_{a_m}}}-\sum_{k=1}^{p}P^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\tfrac{1}{(k-1)!}\Bigg] \\
& +\tfrac{|a|!}{(|a|+p)!}(-1)^{p-1}\tfrac{(p-1)!}{0!}\Bigg[L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+P^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}+N^{(1,1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}- \\
& \sum_{i=2}^{p}N^{(i,1)}_{0_{p-i}1_{a_1}2_{a_2}\cdots m_{a_m}}\tfrac{1}{(i-1)!}+R^{N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}}\Bigg] \\
& +\tfrac{(|a|+1)!}{1!(|a|+p)!}(-1)^{p-2}\tfrac{(p-2)!}{0!}\Bigg[L^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+P^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}+N^{(2,1)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}- \\
& \sum_{l=3}^{p}N^{(l,2)}_{0_{p-l}1_{a_1}\cdots m_{a_m}}\tfrac{1}{(l-1)!}+R^{N^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}}\Bigg] \\
& +\tfrac{(|a|+2)!}{2!(|a|+p)!}(-1)^{p-3}\tfrac{(p-3)!}{0!}\Bigg[L^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+P^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+N^{(3,1)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+ \\
& N^{(3,2)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}+R^{N^{(3)}_{0_{p-3}1_{a_1}\cdots m_{a_m}}}\Bigg]+\cdots+ \\
& +\tfrac{(|a|+p-2)!}{(p-2)!(|a|+p)!}(-1)\tfrac{1!}{0!}\left(L^{(p-1)}_{01_{a_1}\cdots m_{a_m}}+P^{(p-1)}_{01_{a_1}\cdots m_{a_m}}+N^{(p-1,1)}_{01_{a_1}\cdots m_{a_m}}+N^{(p-1,2)}_{01_{a_1}\cdots m_{a_m}}+R^{N^{(p-1)}_{01_{a_1}\cdots m_{a_m}}}\right) \\
& +\tfrac{(|a|+p-1)!}{(p-1)!(|a|+p)!}\left(P^{(p)}_{1_{a_1}\cdots m_{a_m}}+N^{(p,1)}_{1_{a_1}\cdots m_{a_m}}+N^{(p,2)}_{1_{a_1}\cdots m_{a_m}}+R^{N^{(p)}_{1_{a_1}\cdots m_{a_m}}}\right).
\end{aligned}
$$

As a result,

$\sum_{k=-1}^{p} c^{(k)}_{0_{p-k}1_{a_1}\cdots m_{a_m}}\frac{1}{k!}$

$= c^{(-1)}_{0_{p+1}1_{a_1}\cdots m_{a_m}}$
$+ \sum_{k=0}^{p} \frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k+1}\frac{p!(p+|a|)}{(p-k)!} c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}$
$+ \sum_{k=0}^{p} \frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k}\frac{p!}{(p-k)!}\left[-N^{(1,1)}_{0_{p-1}1_{a_1}\cdots m_{a_m}} - \sum_{l=1}^{p-1} L^{(l)}_{0_{p-l}1_{a_1}\cdots m_{a_m}}\frac{1}{(l-1)!} - \sum_{j=1}^{p} R^{N^{(j)}_{0_{p-j}1_{a_1}\cdots m_{a_m}}}\frac{1}{(j-1)!} - \right.$
$\left. \sum_{h=1}^{p} P^{(h)}_{0_{p-h}1_{a_1}\cdots m_{a_m}}\frac{1}{(h-1)!}\right]$
$+ \sum_{k=1}^{p} \frac{|a|!}{(|a|+k)!}(-1)^{k-1}\frac{(p-1)!}{(p-k)!}\left[L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}} + P^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}} + N^{(1,1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}} - \right.$
$\left. \sum_{i=2}^{p} N^{(i,1)}_{0_{p-i}1_{a_1}2_{a_2}\cdots m_{a_m}}\frac{1}{(i-1)!} + R^{N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}}\right]$
$+ \sum_{k=2}^{p} \frac{(|a|+1)!}{1!(|a|+k)!}(-1)^{k-2}\frac{(p-2)!}{(p-k)!}\left[L^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}} + P^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}} + N^{(2,1)}_{0_{p-2}1_{a_1}\cdots m_{a_m}} - \right.$
$\left. \sum_{l=3}^{p} N^{(l,2)}_{0_{p-l}1_{a_1}\cdots m_{a_m}}\frac{1}{(l-1)!} + R^{N^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}}\right]$
$+ \sum_{i=3}^{p-1}\sum_{k=i}^{p} \frac{(|a|+i-1)!}{(i-1)!(|a|+k)!}(-1)^{k-i}\frac{(p-i)!}{(p-k)!}\left[L^{(i)}_{0_{p-i}1_{a_1}\cdots m_{a_m}} + P^{(i)}_{0_{p-i}1_{a_1}\cdots m_{a_m}} + \right.$
$\left. N^{(i,1)}_{0_{p-i}1_{a_1}\cdots m_{a_m}} + N^{(i,2)}_{0_{p-i}1_{a_1}\cdots m_{a_m}} + R^{N^{(i)}_{0_{p-i}1_{a_1}\cdots m_{a_m}}}\right]$
$+ \cdots$
$+ \sum_{k=p}^{p} \frac{(|a|+p-1)!}{(p-1)!(|a|+k)!}(-1)^{k-p}\frac{(p-p)!}{(p-k)!}\left(P^{(p)}_{1_{a_1}\cdots m_{a_m}} + N^{(p,1)}_{1_{a_1}\cdots m_{a_m}} + N^{(p,2)}_{1_{a_1}\cdots m_{a_m}} + R^{N^{(p)}_{1_{a_1}\cdots m_{a_m}}}\right) + R$
$= \left(1 + \sum_{k=0}^{p} \frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k+1}\frac{p!(p+|a|)}{(p-k)!}\right) c^{(-1)}_{0_{p+1}1_{a_1}2_{a_2}\cdots m_{a_m}}$ or
$+ \left(L^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}} + P^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}} + N^{(1,1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}} + R^{N^{(1)}_{0_{p-1}1_{a_1}2_{a_2}\cdots m_{a_m}}}\right)\times$
$\left[\sum_{k=0}^{p} \frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k+1}\frac{p!}{(p-k)!} - \sum_{k=1}^{p}\frac{|a|!}{(|a|+k)!}(-1)^{k}\frac{(p-1)!}{(p-k)!}\right]$
$+ \left(L^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}} + P^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}} + R^{N^{(2)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}}\right)\times$
$\left[\sum_{k=0}^{p} \frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k+1}\frac{p!}{(p-k)!} - \sum_{k=2}^{p}\frac{(|a|+1)!}{1!(|a|+k)!}(-1)^{k-1}\frac{(p-2)!}{(p-k)!}\right]$
$+ N^{(2,1)}_{0_{p-2}1_{a_1}\cdots m_{a_m}}\left[\sum_{k=1}^{p} \frac{|a|!}{(|a|+k)!}(-1)^{k}\frac{(p-1)!}{(p-k)!} - \sum_{k=2}^{p}\frac{(|a|+1)!}{1!(|a|+k)!}(-1)^{k-1}\frac{(p-2)!}{(p-k)!}\right]$
$+ \sum_{i=3}^{p-1}\left(L^{(i)}_{0_{p-i}1_{a_1}\cdots m_{a_m}} + P^{(i)}_{0_{p-i}1_{a_1}\cdots m_{a_m}} + R^{N^{(i)}_{0_{p-i}1_{a_1}\cdots m_{a_m}}}\right)\times$
$\left[\sum_{k=0}^{p} \frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k+1}\frac{p!}{(p-k)!}\frac{1}{(i-1)!} - \sum_{k=i}^{p}\frac{(|a|+i-1)!}{(i-1)!(|a|+k)!}(-1)^{k+1-i}\frac{(p-i)!}{(p-k)!}\right]$
$+ \sum_{i=3}^{p} N^{(i,1)}_{0_{p-i}1_{a_1}\cdots m_{a_m}}\left[\sum_{k=1}^{p} \frac{|a|!}{(|a|+k)!}(-1)^{k}\frac{(p-1)!}{(p-k)!}\frac{1}{(i-1)!} - \sum_{k=i}^{p}\frac{(|a|+i-1)!}{(i-1)!(|a|+k)!}(-1)^{k+1-i}\frac{(p-i)!}{(p-k)!}\right]$
$+ \sum_{i=3}^{p} N^{(i,2)}_{0_{p-i}1_{a_1}\cdots m_{a_m}}\left[\sum_{k=2}^{p} \frac{(|a|+1)!}{1!(|a|+k)!}(-1)^{k-1}\frac{(p-2)!}{(p-k)!}\frac{1}{(i-1)!} - \sum_{k=i}^{p}\frac{(|a|+i-1)!}{(i-1)!(|a|+k)!}(-1)^{k+1-i}\frac{(p-i)!}{(p-k)!}\right]$
$+ \left(P^{(p)}_{1_{a_1}\cdots m_{a_m}} + R^{N^{(p)}_{1_{a_1}\cdots m_{a_m}}}\right)\left[\sum_{k=0}^{p} \frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k+1}\frac{p!}{(p-k)!}\frac{1}{(i-1)!} - \sum_{k=p}^{p}\frac{(|a|+p-1)!}{(p-1)!(|a|+k)!}(-1)^{k+1-p}\frac{(p-p)!}{(p-k)!}\right] +$
R

$= R_{0_{p+1}1_{a_1}\cdots m_{a_m}}$ since

$\sum_{k=i}^{p}\frac{(|a|+i-1)!}{(i-1)!(|a|+k)!}(-1)^{k+1-i}\frac{(p-i)!}{(p-k)!} = -\frac{1}{p+|a|}\frac{1}{(i-1)!}$ for all i, $1 \le i \le p$ and a non-negative integer $|a|$ because of Lemma A4 and the fact that $\sum_{k=0}^{p}\frac{|a|!}{(|a|+k)!}\frac{1}{|a|}(-1)^{k+1}\frac{p!}{(p-k)!} = -\frac{1}{|a|} - p\frac{1}{|a|}\sum_{k=1}^{p}\frac{|a|!}{(|a|+k)!}(-1)^{k}\frac{(p-1)!}{(p-k)!}$
$= -\frac{1}{|a|} - \frac{p}{|a|}\left(-\frac{1}{p+|a|}\right) = -\frac{1}{|a|}\left[1 - \frac{p}{p+|a|}\right] = -\frac{1}{|a|}\left[\frac{|a|}{p+|a|}\right] = -\frac{1}{p+|a|}$. ■

Lemma A6 *The coefficient of* Δ^{a_0} *in* $\widetilde{l}^{(\infty)}_X(t, x \mid t_0, x_0)$ *is free from the indeterminate terms of* $C^{(j_{-1},-1)}_X$ *for all* $a_0 \ge 1$.

Proof. As can be seen from the proof of Lemma A1, the $c_{0_p}^{(0)}(t_0, x_0)$, $p \geq 1$, terms are cancelled out by $C_X^{(j_k,k)}$, $1 \leq k \leq p$ in $\widetilde{l}_X^{(\infty)}(t, x \mid t_0, x_0)$. Hence, we can focus on the coefficients of Δ^p, $p \geq 1$, that are dependent on the indeterminate terms of $C_X^{(j_{-1},-1)}$ in each $C_X^{(j_k.k)}$, $k \neq 0$, in order to prove this theorem.

First consider the case $a_0 = 1$. The Δ term of $\widetilde{l}_X^{(\infty)}$ comes from $\frac{c_{0_2}^{(-1)}\Delta^2}{\Delta} + c^{(1)}\Delta$, which can be shown to not have indeterminacy problem analogously to the proof below and details are excluded. One difference is that $N^{(1,2)}$ term is not associated in this case.

Now consider the coefficient of Δ^p, $p \geq 2$ in $\widetilde{l}_X^{(\infty)}(t, x \mid t_0, x_0)$. This term is obtained from $\frac{c_{0_{p+1}}^{(-1)}\Delta^{p+1}}{\Delta} + \sum_{k=1}^{p} c_{0_{p-k}}^{(k)}\Delta^{p-k}\frac{\Delta^k}{k!}$.

Using $\widetilde{f}_X^{(-2)}(t, x|t_0, x_0) = 0$, $c_{0_{p+1}}^{(-1)}$ satisfies $-2c_{0_{p+1}}^{(-1)}\Delta^{p+1} = 2Q_{0_{p+1}}^{(-1)}\Delta^{p+1}$, where
$Q_{0_{p+1}}^{(-1)}\Delta^{p+1} = \frac{1}{2}\sum_{\substack{e_0=1\\e_0+b_0+d_0=p+1}}^{p}\sum_{b_0=1}^{p}\sum_{d_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{d_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{d_0}}\frac{\partial c_{0_{b_0}i}^{(-1)}\Delta^{b_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial c_{0_{e_0}j}^{(-1)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$ so

$$-Q_{0_{p+1}}^{(-1)} = c_{0_{p+1}}^{(-1)}. \tag{65}$$

The Δ^p term in $\widetilde{f}_X^{(-1)}(t, x|t_0, x_0) = 0$ is $-L_{0_p}^{(0)}\Delta^p = -\frac{\partial c_{0_{p+1}}^{(-1)}\Delta^{p+1}}{\partial t} + P_{0_p}^{(0)}\Delta^p + R\Delta^p$, where
$L_{0_p}^{(0)}\Delta^p = \sum_{\substack{e_0=0\\e_0+b_0+d_0=p}}^{p-1}\sum_{b_0=1}^{p}\sum_{d_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{d_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{d_0}}\frac{\partial c_{0_{b_0}i}^{(-1)}\Delta^{b_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial c_{0_{e_0}j}^{(0)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$ and
$P_{0_p}^{(0)}\Delta^p = -\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{i=1}^{m}\frac{\partial^{b_0}\mu_i(t_0,x_0)}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}i}^{(-1)}\Delta^{e_0}(x_i-x_{0i})}{\partial x_i} + \sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)}{\partial x_i^{b_0}}\frac{\partial c_{0_{e_0}j}^{(-1)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$
$-\sum_{\substack{e_0=1\\e_0+b_0+d_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}i}^{(-1)}\Delta^{e_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial^{d_0}\left[\frac{\partial D_\upsilon(t_0,x_0)}{\partial x_j}\right]\Delta^{d_0}}{\partial t^{d_0}} +$
$\frac{1}{2}\sum_{\substack{e_0=1\\e_0+b_0=p}}^{p}\sum_{b_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}ij}^{(-1)}\Delta^{e_0}(x_i-x_{0i})(x_j-x_{0j})}{\partial x_i\partial x_j}$. Consequently

$$L_{0_p}^{(0)} = (p+1)\,c_{0_{p+1}}^{(-1)} - P_{0_p}^{(0)} + R. \tag{66}$$

Recall that we can ignore $c_{0_p}^{(0)}$ because of Lemma A1. The equation for $c_{0_{p-1}}^{(1)}$ can be found from $\widetilde{f}_X^{(0)}(t, x|t_0, x_0) = 0$ as

$c_{0_{p-1}}^{(1)}\Delta^{p-1} - L_{0_{p-1}}^{(1)}\Delta^{p-1} = P_{0_{p-1}}^{(1)}\Delta^{p-1} + N_{0_{p-1}}^{(1)}\Delta^{p-1} + R\Delta^{p-1}$. Here
$L_{0_{p-1}}^{(1)}\Delta^{p-1} = \sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-2}\sum_{b_0=1}^{p-1}\sum_{d_0=0}^{p-2}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{d_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{d_0}}\frac{\partial c_{0_{b_0}i}^{(-1)}\Delta^{b_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial c_{0_{e_0}j}^{(1)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$,
$P_{0_{p-1}}^{(1)}\Delta^{p-1} = -\sum_{\substack{e_0=0\\e_0+b_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{i=1}^{m}\frac{\partial^{b_0}\mu_i(t_0,x_0)}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}i}^{(0)}\Delta^{e_0}(x_i-x_{0i})}{\partial x_i} + \sum_{\substack{e_0=0\\e_0+b_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)}{\partial x_i^{b_0}}\frac{\partial c_{0_{e_0}j}^{(0)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$
$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}i}^{(0)}\Delta^{e_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial^{d_0}\left[\frac{\partial D_\upsilon(t_0,x_0)}{\partial x_j}\right]\Delta^{d_0}}{\partial t^{d_0}} +$
$\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}ij}^{(0)}\Delta^{e_0}(x_i-x_{0i})(x_j-x_{0j})}{\partial x_i\partial x_j}$ and

$N_{0_{p-1}}^{(1)}\Delta^{p-1}=\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-1}}^{p-1}\sum_{b_0=0}^{p-1}\sum_{d_0=0}^{p-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{d_0}\upsilon_{ij}(t_0,x_0)}{\partial t^{d_0}}\frac{\partial c_{0_{b_0}i}^{(0)}\Delta^{b_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial c_{0_{e_0}j}^{(0)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$. Therefore

$$c_{0_{p-1}}^{(1)}-L_{0_{p-1}}^{(1)}=P_{0_{p-1}}^{(1)}+N_{0_{p-1}}^{(1)}+R. \tag{67}$$

Consider the term Δ^{p-k} in $\widetilde{f}_X^{(k-1)}(t,x|t_0,x_0)=0,\ 2\leq k\leq p$, then
$c_{0_{p-k}}^{(k)}\Delta^{p-k}-L_{0_{p-k}}^{(k)}\Delta^{p-k}=-\frac{\partial c_{0_{p-k+1}}^{(k-1)}\Delta^{p-k+1}}{\partial t}+P_{0_{p-k}}^{(k)}\Delta^{p-k}+N_{0_{p-k}}^{(k)}\Delta^{p-k}+R\Delta^{p-k}$. Here
$L_{0_{p-k}}^{(k)}\Delta^{p-k}=\frac{1}{k}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k-1}\sum_{b_0=1}^{p-k}\sum_{d_0=0}^{p-k-1}\sum_{i=1}^{m}\sum_{j=1}^{m}\left[\frac{\partial^{d_0}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}}{\partial t^{d_0}}\frac{\partial c_{0_{b_0}i}^{(-1)}\Delta^{b_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial c_{0_{e_0}j}^{(k)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}\right]$.

Note that $P_{0_{p-k}}^{(k)}\Delta^{p-k}=\sum_{\substack{e_0=0\\e_0+b_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{i=1}^{m}\frac{\partial^{b_0}\mu_i(t_0,x_0)\Delta^{b_0}}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}i}^{(k-1)}\Delta^{e_0}(x_i-x_{0i})}{\partial x_i}+$

$\sum_{\substack{e_0=0\\e_0+b_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}}{\partial t^{b_0}}\left[\frac{\partial\upsilon_{ij}(t_0,x_0)}{\partial x_i}\right]\Delta^{b_0}\frac{\partial c_{0_{e_0}j}^{(k-1)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$

$-\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}}{\partial t^{b_0}}\frac{\partial c_{0_{e_0}i}^{(k-1)}\Delta^{e_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial^{d_0}}{\partial t^{d_0}}\left[\frac{\partial D_\upsilon(t_0,x_0)}{\partial x_j}\right]\Delta^{d_0}$

$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{b_0}\upsilon_{ij}(t_0,x_0)\Delta^{b_0}}{\partial^{b_0}}\frac{\partial c_{0_{e_0}ij}^{(k-1)}\Delta^{e_0}(x_i-x_{0i})(x_j-x_{0j})}{\partial x_i\partial x_j}$ and

$N_{0_{p-k}}^{(k)}\Delta^{p-k}=$

$\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{d_0}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}}{\partial t^{d_0}}\frac{\partial c_{0_{b_0}i}^{(0)}\Delta^{b_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial c_{0_{e_0}j}^{(k-1)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}$

$+\frac{1}{2}\sum_{\substack{e_0=0\\e_0+b_0+d_0=p-k}}^{p-k}\sum_{b_0=0}^{p-k}\sum_{d_0=0}^{p-k}\sum_{i=1}^{m}\sum_{j=1}^{m}\frac{\partial^{d_0}\upsilon_{ij}(t_0,x_0)\Delta^{d_0}}{\partial t^{d_0}}\left[\sum_{h=1}^{k-2}\binom{k-1}{h}\frac{\partial c_{0_{b_0}i}^{(h)}\Delta^{b_0}(x_i-x_{0i})}{\partial x_i}\frac{\partial c_{0_{e_0}j}^{(k-1-h)}\Delta^{e_0}(x_j-x_{0j})}{\partial x_j}\right]$

that are the Δ^{p-k} terms from $\widetilde{G}_X^{(k,1)}(t,x\mid t_0,x_0)$ and $\widetilde{G}_X^{(k,2)}(t,x\mid t_0,x_0)$, respectively, for $k\geq 2$. Accordingly,

$$c_{0_{p-k}}^{(k)}-L_{0_{p-k}}^{(k)}=-(p-k+1)\,c_{0_{p-k+1}}^{(k-1)}+P_{0_{p-k}}^{(k)}+N_{0_{p-k}}^{(k)}+R. \tag{68}$$

Using the fact that $c_{0_{e_0}i}^{(-1)}=-\sum_{k=0}^{e_0-1}c_{0_{e_0-k-1}i}^{(k)}\frac{1}{k!}+R_{0_{e_0}i}$ and $c_{0_{b_0}i}^{(0)}=-c_{0_{b_0+1}i}^{(-1)}-\sum_{l=1}^{b_0}c_{0_{b_0-l}i}^{(l)}\frac{1}{l!}+R_{0_{b_0+1}i}$ from Lemma A2, A3 and A5, similarly to the proof of Lemma A5, we get

$$P_{0_p}^{(0)}=-\sum_{k=1}^{p}P_{0_{p-k}}^{(k)}\frac{1}{(k-1)!}+R, \tag{69}$$

for $1\leq k\leq p$,

$$N_{0_{p-k}}^{(k)}=N_{0_{p-k}}^{(k,1)}+N_{0_{p-k}}^{(k,2)}+R^{N_{0_{p-k}}^{(k)}}+R, \tag{70}$$

$$N_{0_{p-1}}^{(1,2)}=-\sum_{k=2}^{p}N_{0_{p-k}}^{(k,1)}\frac{1}{(k-1)!}+R, \tag{71}$$

$$\sum_{k=2}^{p} N_{0_{p-k}}^{(k,2)} \frac{1}{(k-1)!} = 0, \tag{72}$$

$$L_{0_p}^{(0)} = -Q_{0_{p+1}}^{(-1)} - N_{0_{p-1}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} + R^{L_{0_p}^{(0)}}. \tag{73}$$

and

$$R^{L_{0_p}^{(0)}} = -\sum_{k=1}^{p} R^{N_{0_{p-k}}^{(k)}} \frac{1}{(k-1)!} + R. \tag{74}$$

We are ready to show that $c_{0_{p+1}}^{(-1)} + \sum_{k=1}^{p} c_{0_{p-k}}^{(k)} \frac{1}{k!} = R_{0_{p+1}}$. First, using (65) and (74), (73) becomes

$$L_{0_p}^{(0)} = c_{0_{p+1}}^{(-1)} - N_{0_{p-1}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} - \sum_{k=1}^{p} R^{N_{0_{p-k}}^{(k)}} \frac{1}{(k-1)!}$$

and (66) is equivalent to

$$L_{0_p}^{(0)} = (p+1)\, c_{0_{p+1}}^{(-1)} + \sum_{k=1}^{p} P_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} + R$$

due to (69). Combining these two

$$c_{0_{p+1}}^{(-1)} = \frac{1}{p}\left[-\sum_{k=1}^{p} P_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} - N_{0_{p-1}}^{(1,1)} - \sum_{k=1}^{p-1} L_{0_{p-k}}^{(k)} \frac{1}{(k-1)!} - \sum_{k=1}^{p} R^{N_{0_{p-k}}^{(k)}} \frac{1}{(k-1)!}\right].$$

From (67),

$$c_{0_{p-1}}^{(1)} = L_{0_{p-1}}^{(1)} + P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1,1)} - \sum_{k=2}^{p} N_{0_{p-k}}^{(k,1)} \frac{1}{(k-1)!} + R^{N_{0_{p-1}}^{(1)}} + R$$

because of (70) and (71). Next,

$$c_{0_{p-2}}^{(2)} = L_{0_{p-2}}^{(2)} + P_{0_{p-2}}^{(2)} + N_{0_{p-2}}^{(2,1)} - \sum_{k=3}^{p} N_{0_{p-k}}^{(k,2)} \frac{1}{(k-1)!} + R^{N_{0_{p-2}}^{(2)}}$$

for the fact (70) and (72).

Because, for $k \geq 2$,

$$\begin{aligned}
&c_{0_{p-k}}^{(k)} = -(p-k+1)\, c_{0_{p-k+1}}^{(k-1)} + L_{0_{p-k}}^{(k)} + P_{0_{p-k}}^{(k)} + N_{0_{p-k}}^{(k)} + R \\
&= (-1)^{k-1} \tfrac{(p-1)!}{(p-k)!} \left[L_{0_{p-1}}^{(1)} + P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1)}\right] + (-1)^{k-2} \tfrac{(p-2)!}{(p-k)!} \left[L_{0_{p-2}}^{(2)} + P_{0_{p-2}}^{(2)} + N_{0_{p-2}}^{(2)}\right] \\
&+ (-1)^{k-3} \tfrac{(p-3)!}{(p-k)!} \left[L_{0_{p-3}}^{(3)} + P_{0_{p-3}}^{(3)} + N_{0_{p-3}}^{(3)}\right] + \cdots + (-1) \tfrac{(p-k+1)!}{(p-k)!} \left[L_{0_{p-k+1}}^{(k-1)} + P_{0_{p-k+1}}^{(k-1)} + N_{0_{p-k+1}}^{(k-1)}\right] \\
&+ L_{0_{p-k}}^{(k)} + P_{0_{p-k}}^{(k)} + N_{0_{p-k}}^{(k)} + R, \\
&c_{0_{p+1}}^{(-1)} + \textstyle\sum_{k=1}^{p} c_{0_{p-k}}^{(k)} \frac{1}{k!} = \frac{1}{p}\left[-\sum\limits_{h=1}^{p} P_{0_{p-h}}^{(h)} \frac{1}{(h-1)!} - N_{0_{p-1}}^{(1,1)} - \sum\limits_{h=1}^{p-1} L_{0_{p-h}}^{(h)} \frac{1}{(h-1)!} - \sum\limits_{h=1}^{p} R^{N_{0_{p-h}}^{(h)}} \frac{1}{(h-1)!}\right] \\
&+ \textstyle\sum_{k=1}^{p} (-1)^{k-1} \frac{(p-1)!}{(p-k)!} \left[L_{0_{p-1}}^{(1)} + P_{0_{p-1}}^{(1)} + N_{0_{p-1}}^{(1,1)} - \sum\limits_{h=2}^{p} N_{0_{p-h}}^{(h,1)} \frac{1}{(h-1)!} + R^{N_{0_{p-1}}^{(1)}}\right] \frac{1}{k!} \\
&+ \textstyle\sum_{k=2}^{p} (-1)^{k-2} \frac{(p-2)!}{(p-k)!} \left[L_{0_{p-2}}^{(2)} + P_{0_{p-2}}^{(2)} + N_{0_{p-2}}^{(2,1)} - \sum\limits_{h=3}^{p} N_{0_{p-h}}^{(h,2)} \frac{1}{(h-1)!} + R^{N_{0_{p-2}}^{(2)}}\right] \frac{1}{k!} \\
&+ \textstyle\sum_{k=3}^{p} (-1)^{k-3} \frac{(p-3)!}{(p-k)!} \left[L_{0_{p-3}}^{(3)} + P_{0_{p-3}}^{(3)} + N_{0_{p-3}}^{(3,1)} + N_{0_{p-3}}^{(3,2)} + R^{N_{0_{p-3}}^{(3)}}\right] \frac{1}{k!} + \cdots + \\
&+ \textstyle\sum_{k=p-1}^{p} (-1)^{k-p+1} \frac{(p-p+1)!}{(p-k)!} \left[L_0^{(p-1)} + P_0^{(p-1)} + N_0^{(p-1,1)} + N_0^{(p-1,2)} + R^{N_0^{(p-1)}}\right] \frac{1}{k!}
\end{aligned}$$

$+\sum_{k=p}^{p}(-1)^{k-p}\frac{(p-p)!}{(p-k)!}\left[L^{(p)}+P^{(p)}+N^{(p,1)}+N^{(p,2)}+R^{N^{(p)}}\right]\frac{1}{k!}$
$=\left[L_{0_{p-1}}^{(1)}+P_{0_{p-1}}^{(1)}+N_{0_{p-1}}^{(1,1)}+R^{N_{0_{p-1}}^{(1)}}\right]\left(-\frac{1}{p}+\sum_{k=1}^{p}(-1)^{k-1}\frac{(p-1)!}{(p-k)!}\frac{1}{k!}\right)$
$+\left[L_{0_{p-2}}^{(2)}+P_{0_{p-2}}^{(2)}+R^{N_{0_{p-2}}^{(2)}}\right]\left(-\frac{1}{p}+\sum_{k=2}^{p}(-1)^{k-2}\frac{(p-2)!}{(p-k)!}\frac{1}{k!}\right)$
$+\sum_{i=3}^{p-1}\left(L_{0_{p-i}}^{(i)}+P_{0_{p-i}}^{(i)}+R^{N_{0_{p-i}}^{(i)}}\right)\left(-\frac{1}{p}\frac{1}{(i-1)!}+\sum_{k=i}^{p}(-1)^{k-i}\frac{(p-i)!}{(p-k)!}\frac{1}{k!}\right)$
$+\sum_{i=2}^{p}N_{0_{p-i}}^{(i,1)}\left(-\sum_{k=1}^{p}(-1)^{k-1}\frac{(p-1)!}{(p-k)!}\frac{1}{k!}\frac{1}{(i-1)!}+\sum_{k=i}^{p}(-1)^{k-i}\frac{(p-i)!}{(p-k)!}\frac{1}{k!}\right)$
$+\sum_{i=3}^{p}N_{0_{p-i}}^{(i,2)}\left(-\sum_{k=2}^{p}(-1)^{k-2}\frac{(p-2)!}{(p-k)!}\frac{1}{k!}\frac{1}{(i-1)!}+\sum_{k=i}^{p}(-1)^{k-i}\frac{(p-i)!}{(p-k)!}\frac{1}{k!}\right)$
$+\left[P^{(p)}+R^{N^{(p)}}\right]\left(-\frac{1}{p}\frac{1}{(p-1)!}+\frac{1}{p!}\right)+R=R_{0_{p+1}}$ due to Lemma A4 where $m=0$. ■

Proof of Theorem 3

Proof. The constant term of $\widetilde{l}_X^{(\infty)}(t,x\mid t_0,x_0)$ is zero because the constant and first order terms of $C_X^{(j_{-1},-1)}$ and the constant term of $C_X^{(j_0,0)}$ are zero. Combining Lemma A1, Lemma A2, Lemma A3, Lemma A5, and Lemma A6, it is clear that all terms in $\widetilde{l}_X^{(\infty)}(t,x\mid t_0,x_0)$ are indeterminate terms free. ■

Proof of Theorem 4

Proof. We will prove the irreducible case. Removing additional notations for Taylor expansion in x from this proof gives the proof of the reducible case. Taylor-expanding $\exp\left(\widetilde{l}_X^{(K)}\right)$ in (t,x) about (t_0,x_0) yields $\widetilde{p}_X^{(K)}$, which satisfies the Kolmogorov backward equation

$$-\frac{\partial p_X(t,x\mid t_0,x_0)}{\partial t_0}=\sum_{i=1}^{m}\mu_i(t_0,x_0)\frac{\partial p_X(t,x\mid t_0,x_0)}{\partial x_{0i}}+\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\upsilon_{ij}(t_0,x_0)\frac{\partial^2 p_X(t,x\mid t_0,x_0)}{\partial x_{0i}\partial x_{0j}}. \tag{75}$$

Because the coefficients $C_X^{(j_k,k)}$, $k=-1,0,\cdots,K$ are found after matching the same order terms of left- and right-hand sides of (75) so that its approximation error can be $O\left(\Delta^{K+1}\right)$,

$$\begin{aligned}\widetilde{B}_X^{(K)}(t,x\mid t_0,x_0) &= -\frac{\partial\widetilde{p}_X^{(K)}(t,x\mid t_0,x_0)}{\partial t_0}-\sum_{i=1}^{m}\widetilde{\mu}_i(t_0,x_0)\frac{\partial\widetilde{p}_X^{(K)}(t,x\mid t_0,x_0)}{\partial x_{0i}}\\ &\quad -\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\widetilde{\upsilon}_{ij}(t_0,x_0)\frac{\partial^2\widetilde{p}_X^{(K)}(t,x\mid t_0,x_0)}{\partial x_{0i}\partial x_{0j}}\end{aligned} \tag{76}$$

is the remainder term. In fact $\widetilde{B}_X^{(K)}(t,x\mid t_0,x_0)=\Delta^K\widetilde{p}_X^{(K)}(t,x\mid t_0,x_0)\widetilde{\psi}_X^{(K)}(t,x\mid t_0,x_0)$ where $\widetilde{\psi}_X^{(K)}(t,x\mid t_0,x_0)$ is a sum of products of $\widetilde{\mu}_i$, $\widetilde{\upsilon}_{ij}$, $C_X^{(j_k,k)}$, $k\geq -1$ and their derivatives. $\widetilde{\psi}_X^{(K)}(t,x\mid t_0,x_0)$ has at most polynomial growth because $C_X^{(j_k,k)}$ is determined by $\widetilde{\mu}_i$ and $\widetilde{\sigma}_{ij}$ that exhibit polynomial growth due to Assumption 4. $\widetilde{\psi}_X^{(K)}(t,x\mid t_0,x_0)=O(1)$ uniformly for all (t_0,x_0) and (t,x) in a compact subset of the interior of $[0,\infty)\times S_X$ and for all θ in Θ for the reason that μ_i, σ_{ij}, and their derivatives are continuous.

Define $\widetilde{r}_X^{(K)}(t,x|t_0,x_0)=\widetilde{p}_X^{(K)}(t,x|t_0,x_0)-p_X(t,x|t_0,x_0)$ then $\widetilde{r}_X^{(K)}(t,x|t_0,x_0)$ also satisfies (75) with the same remainder $\widetilde{B}_X^{(K)}(t,x\mid t_0,x_0)$ since (75) holds true for $p_X(t,x|t_0,x_0)$. In addition, $\widetilde{r}_X^{(K)}(t,x|t_0,x_0)\rightarrow 0$

as $\Delta \to 0$ on account of the fact that both $\widetilde{p}_X^{(K)}(t, x|t_0, x_0)$ and $p_X(t, x|t_0, x_0)$ converge to a Dirac mass at (t_0, x_0) as $\Delta \to 0$. As a matter of fact

$$\widetilde{r}_X^{(K)}(t, x|t_0, x_0) = \int_{S_X} \int_t^{t_0} \widetilde{B}_X^{(K)}(t, x \mid \tau, z)\, p_X(0, z|\tau - t_0, x_0)\, d\tau dz \tag{77}$$

is the solution because (77) satisfies the initial boundary condition and (76). As can be seen below, the tails of p_X decrease at an exponential rate in a neighborhood of $\Delta = 0$ while $\widetilde{B}_X^{(K)}$ has polynomial growth. Hence $\widetilde{B}_X^{(K)} p_X$ is integrable and $\widetilde{r}_X^{(K)}(t, x|t_0, x_0) = O(\Delta^K)$ uniformly for all (t_0, x_0) and (t, x) in a compact subset of the interior of $[0, \infty) \times S_X$ and for all θ in Θ. Let

$$\widetilde{R}_X^{(K)}(t, x|t_0, x_0) \equiv \sup_{\theta \in \Theta} \left| \widetilde{r}_X^{(K)}(t, x|t_0, x_0) \right|.$$

Then

$$\begin{aligned} E\left[\widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \middle| X_0 = x_0 \right] &= \int_{S_X} \widetilde{R}_X^{(K)}(t, x|t_0, x_0)\, p_X(t, x|t_0, x_0)\, dx \\ &= \int_N \widetilde{R}_X^{(K)}(t, x|t_0, x_0)\, p_X(t, x|t_0, x_0)\, dx \\ &\quad + \int_{N \backslash S_X} \widetilde{R}_X^{(K)}(t, x|t_0, x_0)\, p_X(t, x|t_0, x_0)\, dx, \end{aligned}$$

where $N = \prod_{i=1}^m \left[x_{0i} - \sqrt{\Delta} c_\Delta, x_{0i} + \sqrt{\Delta} c_\Delta \right]$ is a neighborhood of x_0, $N \backslash S_X$ is its complement and c_Δ is a sequence of positive numbers such that $c_\Delta \to \infty$ and $\sqrt{\Delta} c_\Delta \to 0$.

Because $\widetilde{r}_X^{(K)}(t, x|t_0, x_0) = O(\Delta^K)$, $\widetilde{R}_X^{(K)}(t, x|t_0, x_0) \leq M \Delta^K$ for some constant M and for all $x \in N$. Thus

$$\int_N \widetilde{R}_X^{(K)}(t, x|t_0, x_0)\, p_X(t, x|t_0, x_0)\, dx = O(\Delta^K).$$

In a neighborhood of $\Delta = 0$, the tail behavior of p_X is driven by the term $\exp\left[-\frac{m}{2} \ln(\Delta) + C_X^{(j_{-1}, -1)}(t, x|t_0, x_0)\, \Delta^{-1} \right]$, with $C_X^{(j_{-1}, -1)}(t, x|t_0, x_0) = -(1/2)(x - x_0)^T v^{-1}(t_0, x_0)(x - x_0) + o(\Delta)$. Because $\widetilde{R}_X^{(K)}$ grows at a polynomial rate, the expected value of $\widetilde{R}_X^{(K)}$ outside of N involves integrating $\|x - x_0\|^b$, $b \geq 0$ against the exponential tails of p_X. Considering the $m = 1$ case for simplicity, on the interval $\left(\sqrt{\Delta} c_\Delta, \infty \right)$, it is of the form

$$\sqrt{\Delta} \int_{\sqrt{\Delta} c_\Delta}^{\infty} |x - x_0|^b \exp\left[-\frac{(x - x_0)^2}{2\Delta v(t_0, x_0)} \right] dx = \Delta^{b/2} \int_{c_\Delta}^{\infty} |z - z_0|^b \exp\left[-\frac{(z - z_0)^2}{2v(t_0, x_0)} \right] dz$$

by the change of variable $z - z_0 = (x - x_0)/\sqrt{\Delta}$ and similarly on the interval $\left(-\infty, -\sqrt{\Delta} c_\Delta \right)$. These integrals converges to zero since $c_\Delta \to \infty$. Hence $E\left[\widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \middle| X_{t_0} = x_0 \right] \to 0$ as $\Delta \to 0$.

Following the similar procedure we can prove that $Var\left[\widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \middle| X_{t_0} = x_0 \right] \to 0$ as $\Delta \to 0$. Combining these two results, given X_{t_0}, $\widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \to 0$ in P_{θ_0}-probability due to the Chebyshev's inequality. For a positive number ε,

$$P\left(\left| \widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \right| > \varepsilon \right) = \int_{S_X} P\left(\left| \widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \right| > \varepsilon \middle| X_{t_0} = x_0 \right) \pi_{t_0}(x_0)\, dx_0,$$

where $\pi_{t_0}(x_0)$ is the marginal density of X_t at time t_0. Notice that $0 \leq P(\cdot) \leq 1$ and the density function $\pi_{t_0}(x_0)$ integrates to one. So, applying Lebesque's dominated convergence theorem $\widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0}) \to 0$ in P_{θ_0}-probability without conditioning on X_{t_0}.

Because of the convergence of $\widetilde{R}_X^{(K)}(t, X_t|t_0, X_{t_0})$ to 0 in P_{θ_0}-probability and the fact that the logarithm is continuous, $\widetilde{l}_X^{(K)}(\theta) \to l_X(\theta)$ in P_{θ_0}-probability and so does $\widetilde{l}_n^{(K)}(\theta)$ to $l_n(\theta)$ for fixed n. As a result, the maximizer of $\widetilde{l}_n^{(K)}(\theta)$, $\hat{\theta}_{n,\Delta}^{(K)} \in \Theta$, exists almost surely since the maximizer $\hat{\theta}_{n,\Delta}$ of $l_n(\theta)$ is assumed to exist. And they are close to each other as $\Delta \to 0$ in the sense that $\hat{\theta}_{n,\Delta}^{(K)} \to \hat{\theta}_{n,\Delta}$ in P_{θ_0}-probability. Taking $\Delta \to 0$ sufficiently fast, the speed at which $\hat{\theta}_{n,\Delta}^{(K)} \to \hat{\theta}_{n,\Delta}$ can be made arbitrarily high for any n. Hence a sequence $\Delta_n \to 0$ can be taken so that (41) is satisfied. $\blacksquare$